I will give you exam relevant questions ahead of time.

You have to submit these online prior to your exam.

Pre-exam solving is semi open book.

You are allowed to use: the lecture videos and anything on the course website, ie. NO external resources.

I will look at this before the exam, and I will ask related questions with the goal of exposing holes in your understanding.

I will give you a letter grade.

This will be cumulative with a slight emphasis on computing integrals.

You can prepare a cheat-sheet (+ maybe a printout of your sol'n)

You should not be spending time looking through long resources (wasted time)

Most important point: Understand the solutions that you submitted.

HW5 is due noon on Wednesday (I'll make this shorter)

The exam is on Wednesday and Thursday (next week)

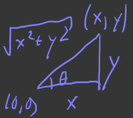
I will release sign-ups tomorrow (first come first serve).

Be punctual. (20 min slots with 5 min gaps)

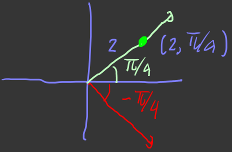
Thursday is a review session.

# Calculus time

We were discussing polar co-ordinates



$(r, \theta)$



$(x, y)$

$$r = \sqrt{x^2+y^2}$$

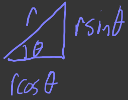
$$\tan(\theta) = \frac{y}{x}$$

$$\cos(\theta) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin(\theta) = \frac{y}{\sqrt{x^2+y^2}}$$

$$x = r\cos(\theta)$$

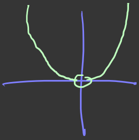
$$y = r\sin(\theta)$$



## Cartesian equations to polar equations

$$y = x^2 \Rightarrow r\sin(\theta) = r^2\cos^2(\theta)$$

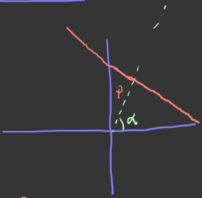
$$\overset{(r \neq 0)}{\Rightarrow} r = \frac{\sin(\theta)}{\cos^2(\theta)} = \tan(\theta)\sec(\theta).$$



This is defined for $0 < \theta < \pi$ except at $\pi/2$ □

$\alpha$ is some fixed number consider

$$x\cos(\alpha) + y\sin(\alpha) = p$$



$$r\cos(\theta)\cos(\alpha) + r\sin(\theta)\sin(\alpha) = p$$

$$\Rightarrow r\cos(\theta - \alpha) = p$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$y = f(x)$

polar analogue: $r = f(\theta)$

$$\Rightarrow r = p\sec(\theta - \alpha) \quad \square$$

$$\boxed{r = 2a\cos(\theta)}$$

$$\Rightarrow \sqrt{x^2+y^2} = 2a\frac{x}{\sqrt{x^2+y^2}}$$

$$\Rightarrow x^2+y^2 = 2ax$$

$$\Rightarrow x^2 - 2ax + y^2 = 0$$

$$\Rightarrow (x-a)^2 - a^2 + y^2 = 0$$

$$\Rightarrow (x-a)^2 + y^2 = a^2 \quad \square$$

$$r = \frac{4}{3+\cos(\theta)}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{4}{3+\frac{x}{\sqrt{x^2+y^2}}}$$

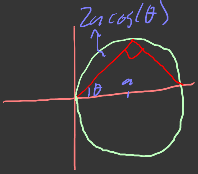
$$\Rightarrow 3\sqrt{x^2+y^2} + x = 4$$

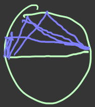
$$\Rightarrow 3\sqrt{x^2+y^2} = 4 - x$$

$$\Rightarrow 9(x^2+y^2) = 16 - 8x + x^2$$

$$(x+b)^2 = x^2 + \underbrace{2b}x + b^2$$

$$(x-a)^2 = x^2 - 2ax + a^2$$





centre: $(a,0)$

radius: $a$

$r = 1$

$$\left(3+\frac{x}{\sqrt{x^2+y^2}}\right)\sqrt{x^2+y^2} = 4$$

$$x^2 + x = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4}$$

ellipse centred at $(-1/2, 0)$

$a = \sqrt{\frac{18}{8}}$

$$\Rightarrow \quad 8x^2 + 8x + 9y^2 = 16$$

$$\Rightarrow \quad 8\left(x+\tfrac{1}{2}\right)^2 - 2 + 9y^2 = 16$$

$$\Rightarrow \quad \boxed{8\left(x+\tfrac{1}{2}\right)^2 + 9y^2 = 18}$$

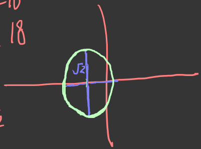
$8a^2 = 18$

$9y^2 = 18$

$y^2 = 2$

$y = \sqrt{2}$

$\approx 1.41$



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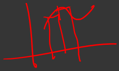
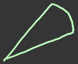
# Area in polar co-ordinates



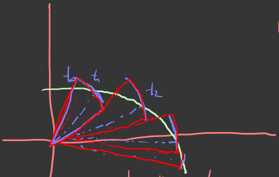


We'll do another hand-waving "physicist argument"

$r = f(\theta)$ for $\theta \in [a,b]$

$a = t_0 < t_1 < \ldots < t_n = b$

$t_i < s_i < t_{i+1}$

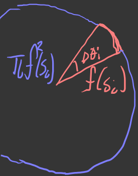


ratio the circle area

$$\frac{\Delta\theta_i}{2\pi}$$

$\Delta\theta_i = t_{i+1} - t_i$

$f(s_i)$



$$\Delta A_i = \pi f^2(s_i)\,\frac{\Delta\theta_i}{2\pi} = \frac{1}{2} f^2(s_i)\,\Delta\theta_i$$

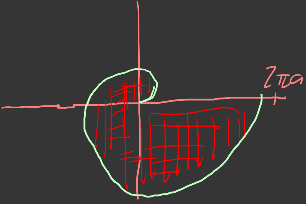
$$A_{aprox} = \sum_{i=0}^{n-1} \frac{1}{2} f^2(s_i)\,\Delta\theta_i$$

$$\Rightarrow \quad \boxed{A = \int_a^b \frac{1}{2} f^2(\theta)\, d\theta}$$

area formula in polar co-ordinates

Example 1:

$r = a\theta$ for $0 \le \theta \le 2\pi$



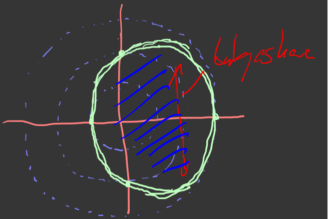
$$A = \int_0^{2\pi} \frac{1}{2} a^2 \theta^2 \, d\theta$$

$$= \frac{1}{6} a^2 \theta^3 \Big|_0^{2\pi} = \frac{1}{6} a^2 \, 8 \pi^3 = \frac{4}{3} a^2 \pi^3 . \square$$

Example 2:

$r = 2 + \cos(\theta)$

for $0 \le \theta \le 2\pi$



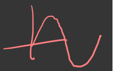
$$A = \int_0^{2\pi} \frac{1}{2} \left(2 + \cos(\theta)\right)^2 d\theta$$

$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$

$$= \int_0^{2\pi} \frac{1}{2} \left(4 + 4\cos(\theta) + \cos^2(\theta)\right) d\theta$$

$$= \int_0^{2\pi} \left[2 + 2\cos(\theta) + \frac{1}{4} + \frac{1}{4}\cos(2\theta)\right] d\theta$$

$$= \frac{9}{4}\theta + 2\sin(\theta) + \frac{1}{8}\sin(2\theta)\Big|_0^{2\pi}$$



$$= \frac{9}{4}2\pi - 0 = \frac{9}{2}\pi.$$

∎

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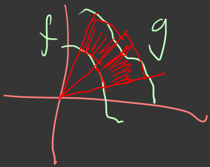
Area between two polar curves:

$r = f(\theta)$, $r = g(\theta)$, $f \geq g \geq 0$.



$$\Delta A_i = \frac{1}{2}\left(f^2(s_i) - g^2(s_i)\right)\Delta\theta_i$$

$$A = \int_a^b \frac{1}{2}\left(f^2(\theta) - g^2(\theta)\right)d\theta$$

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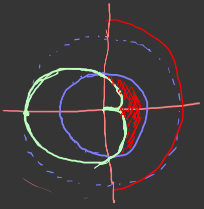
Find the area of the region that lies inside of the circle $r=1$ and outside the "cardioid" $r = 1 - \cos(\theta)$.



They intersect when $-\pi/2 \leq \theta \leq \frac{\pi}{2}$

$$1 - \cos(\theta) = 1 \Rightarrow \cos(\theta) = 0$$

intersection at $\frac{\pi}{2}, -\frac{\pi}{2}$

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1-(1-\cos(\theta))^2\right)d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(2\cos(\theta) - \cos^2(\theta)\right)d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\cos(\theta) - \frac{1}{4} - \frac{1}{4}\cos(2\theta)\right]d\theta$$

$$= \sin(\theta) - \frac{1}{4}\theta - \frac{1}{8}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2}$$

$$= \left(1 - \frac{\pi}{8}\right) - \left(-1 + \frac{\pi}{8}\right) = \boxed{2 - \frac{\pi}{4}}. \blacksquare$$

# Improper integrals:

FTC-I applies to functions (meeting some conditions) defined on all of $[a,b]$.

Even better, integration was only defined for bounded functions on $[a,b]$.

We have no notion at the moment of integration for:

i) unbounded functions e.g. $\int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\,dx$

ii) unbounded ranges e.g. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

We'll presently give meaning to these situations.

## Unbounded ranges:

Suppose $f$ is integrable on $[a,b]$ for all $b \geq a$ (e.g. $f$ is cts.)

then $$\int_a^\infty f(x)\,dx \overset{\text{defn}}{=} \lim_{b\to\infty} \int_a^b f(x)\,dx$$

Similarly if $f$ is int. on $[b,a]$ for all $b \leq a$

then $\int_{-\infty}^{a} f(x)\,dx \overset{\text{defn}}{=} \lim_{b\to-\infty} \int_{b}^{a} f(x)\,dx$

If $f$ is integrable on any $[a,b] \subseteq \mathbb{R}$,

then $\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{defn}}{=} \underbrace{\lim_{b\to-\infty} \int_{b}^{c} f(x)\,dx + \lim_{b\to\infty} \int_{c}^{b} f(x)\,dx}$

↳ both limits must exist

guess: should be $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$
(wrong)

eg, $f(x) = \begin{cases} 0 & \text{for } |x|<1 \\ 1/x & \text{for } |x|\geq 1 \end{cases}$

$b>1$

$$\int_{-b}^{b} f(x)\,dx = \int_{-b}^{-1} 1/x\,dx + \int_{1}^{b} 1/x\,dx$$

$$= -\underbrace{\log(b)}_{-\log|-b|} + \log(b) = 0$$

$$\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx = 0$$

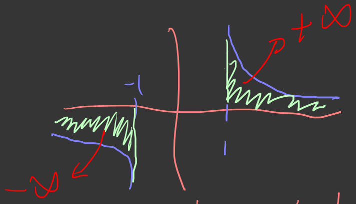
Consider instead

$$\lim_{b\to\infty} \int_{-b}^{2b} f(x)\,dx = \lim_{b\to\infty} \left(-\log(b) + \log(2b)\right)$$

$$= \lim_{b \to \infty} \log\left(\frac{2b}{b}\right) = \underset{b \to \infty}{\log} \log(2) = \log(2)$$

Why? Because $\infty - \infty$ can equal anything

$$\lim_{b \to \infty} \int_{-b+h}^{b+h} f(x)\,dx = \text{something else}$$



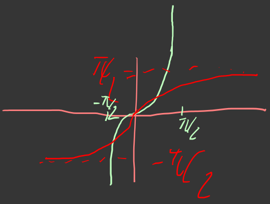
These limits say nothing about area.

We want to avoid $\infty - \infty$.

By splitting it into two limits, we require both limits to independently exist for the integral to converge.

Examples:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$



$$\lim_{b\to-\infty}\int_b^0 \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty} \arctan(x)\Big|_b^0$$

$$= \lim_{b\to-\infty}\left(0-\arctan(b)\right) = \pi/2$$

$$\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \arctan(x)\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\arctan(b)-0\right) = \pi/2$$

Therefore $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \pi/2 + \pi/2 = \pi.$ ∎

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$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \log|x|\Big|_1^b = \lim_{b\to\infty} \log(b)-0$$

$$= \infty$$

ie. This integral $\underline{\text{diverges}}$.

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$$p \neq 1 \quad \int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^b = \lim_{b\to\infty} \frac{1}{1-p}\left(b^{-p+1}-1\right) = (*)$$

If $-p+1<0$ then we have $\lim\limits_{b\to\infty}\frac{1}{b^c}=0$ for $c>0$

$$\lim_{b\to\infty} b^{-p+1} = 0$$

If $-p+1>0$

$$\lim_{b\to\infty} b^{-p+1} = \infty, \quad \text{i.e. DNE}$$

If $1<p$ then

$$(*) = \frac{1}{p-1}\left(1-\lim_{b\to\infty} b^{-p+1}\right) = \frac{1}{p-1}(1+0) = \frac{1}{p-1}$$

If $p<1$ then

$$(*) = \frac{1}{p-1}\left(1 - \lim_{b\to\infty} b^{-p+1}\right) = DNE. \boxtimes$$

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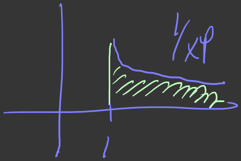
The p-test: (In Calc C, we will use this a lot!)

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & p>1 \\ DNE & p\le 1 \end{cases}$$

This answers the question of when



the shaded area is finite.

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Unbounded functions: Assume $f$ is integrable on every closed subinterval of the relevant domains.

Definitions:

If $f$ is bounded on any closed subinterval of $(a,b]$ Then:

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

$[a, b)$

$$\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$$

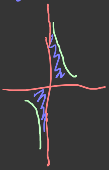
$(a, b]$

Let $d \in (a, b)$

$$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^d f(x)dx + \lim_{c\to b^-}\int_d^c f(x)dx$$

$[a, b] \setminus \{d\}$

e.g. $1/x$ blows up at 0



$$\int_a^b f(x)dx = \lim_{c\to d^-}\int_a^c f(x)dx + \lim_{c\to d^+}\int_c^b f(x)dx$$

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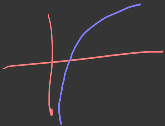
Example $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$



Consider

$$\lim_{b\to 0^-}\int_{-1}^{b} \frac{1}{x}\,dx = \lim_{b\to 0^-} \log|x| \Big|_{-1}^{b}$$

$= \lim_{b \to 0^-} \log|b| = -\infty$ so this limit DNE

Therefore $\int_{-1}^{1} \frac{1}{x}\,dx$ diverges.