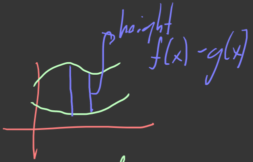
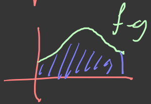
Think like a Physycist.

i.e. intuitive hand-waving justifications



α) Area between the graphs of f and g.
(justified by cavalieri's theorem)



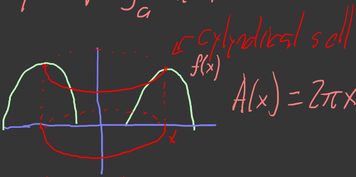
level slices

β) Similarly, finding volumes by slicing.

α) $A = \int_a^b h(x)\,dx$

β) $V = \int_a^b A(x)\,dx$

γ) Cylyndrical shell method

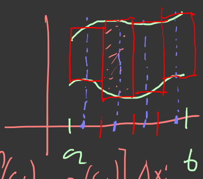


$A(x) = 2\pi x f(x)$

γ) $V = \int_a^b A(x)\,dx$ not level slices

Justifications:

α)



$a = t_0 < t_1 < \dots < t_n = b$

choose points s_i for $i = 0, \dots, n-1$

with $t_i < s_i < t_{i+1}$

$\Delta A_i = [f(s_i) - g(s_i)]\Delta x_i$

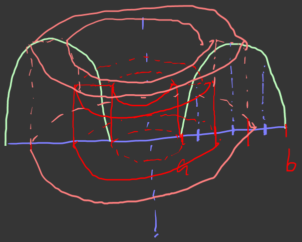
$$A_{aprox} = \sum_{i=0}^{n-1} [f(s_i) - g(s_i)]\Delta x_i$$ where

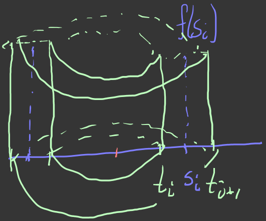
$\Delta t_i = t_{i+1} - t_i$

This should approch the area with finer partitions

$$A = \int_a^b [f(x) - g(x)]\, dx$$

8)





$$\Delta V_i = \pi t_{i+1}^2 f(s_i) - \pi t_i^2 f(s_i) \qquad t_{i+1} - t_i = \Delta t_i$$

$$= \pi (t_{i+1}^2 - t_i^2) f(s_i)$$

$$= \pi (t_{i+1} + t_i)(t_{i+1} - t_i) f(s_i)$$

$$= \pi (2t_i + (t_{i+1} - t_i))(t_{i+1} - t_i) f(s_i)$$

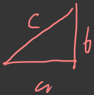
$$= \pi (2t_i + \Delta t_i) \Delta t_i \, f(s_i)$$

$$= 2\pi t_i f(s_i) \Delta t_i + \pi f(s_i) \Delta t_i \Delta t_i$$

$$V_{aprox} = \sum_{i=0}^{n-1} [2\pi t_i f(s_i) + \pi f(s_i) \Delta t_i) \Delta t_i$$

$$V = \int_a^b \left[2\pi x f(x) + \pi f(x) 0\right] dx$$

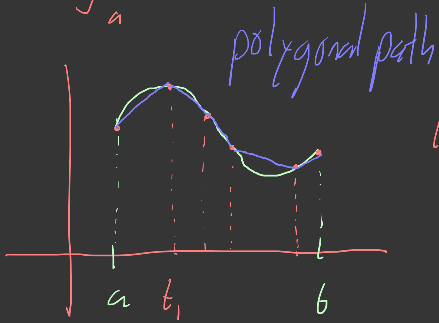
$a^2+b^2=c^2$

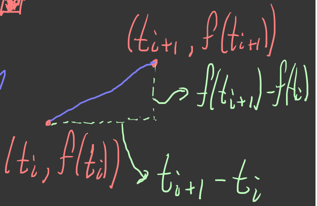


$c=\sqrt{a^2+b^2}$

$$= \int_a^b 2\pi x f(x)\, dx \quad \blacksquare$$

5)





$$\Delta l_i = \sqrt{(t_{i+1}-t_i)^2+(f(t_{i+1})-f(t_i))^2} = \sqrt{1+\left(\frac{f(t_{i+1})-f(t_i)}{t_{i+1}-t_i}\right)^2}\,(t_{i+1}-t_i)$$

$$= \sqrt{1+\left(\frac{f(t_i+\Delta t_i)-f(t_i)}{\Delta t_i}\right)^2}\,\Delta t_i$$

$$l_{aprox} = \sum_{i=0}^{n-1}\sqrt{1+\left(\frac{f(t_i+\Delta t_i)-f(t_i)}{\Delta t_i}\right)^2}\,\Delta t_i$$

$$l = \int_a^b \sqrt{1 + (f'(x))^2}\, dt$$

because:

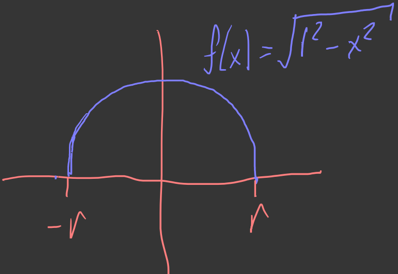
$$\lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{difference quotient}} = f'(x)$$

$$\boxed{\begin{array}{l}\text{Formula for arclength:}\\ l = \int_a^b \sqrt{1 + (f'(x))^2}\, dx\end{array}}$$

Let's use this to find the circumference of a circle.

$$x^2 + y^2 = r^2$$

not a function



$$y = \sqrt{r^2 - x^2}$$

We will find the arclength of $f(x) = \sqrt{r^2 - x^2}$ from $[-r, r]$

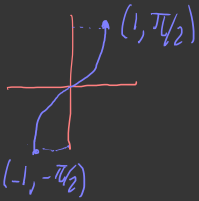
$$f'(x) = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2}$$

$$\ell = \int_{-r}^{r} \sqrt{1 + (f'(x))^2}\, dx = \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$\left[\begin{array}{l} \text{Let } x = r\sin(u) \\ dx = r\cos(u)\, du \end{array}\right] = \int_{\arcsin(-1)}^{\arcsin(1)} \frac{r^2 \cos(u)}{\sqrt{r^2 - r^2 \sin^2(u)}}\, dy$$

$u = \arcsin\left(\frac{x}{r}\right)$



$$= \int_{-\pi/2}^{\pi/2} \frac{r^2 \cos(u)}{r\sqrt{1-\sin^2(u)}}\,du$$

$$= \int_{-\pi/2}^{\pi/2} r\,\frac{\cos(u)}{\cos(u)}\,du = r\int_{-\pi/2}^{\pi/2} 1\,du$$

$$= ru\Big|_{-\pi/2}^{\pi/2} = \pi r$$

$\Rightarrow$ πr is the arclength of the semicircle

$$\Rightarrow \boxed{C = 2\pi r.}$$

∎

Find the arc length of

$f(x) = \frac{1}{3}\left(x^2+2\right)^{3/2}$ over $[0,1]$.

$$f'(x) = \frac{1}{2}\left(x^2+2\right)^{1/2} 2x = x\sqrt{x^2+2}$$

$$1 + \left(f'(x)\right)^2 = 1 + x^2\left(x^2+2\right)$$

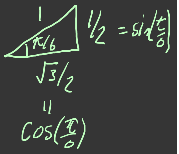
$$= x^4 + 2x^2 + 1 = (x^2+1)^2$$

$$\ell = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 [x^2+1]\, dx$$

$$= \left(\frac{x^3}{3} + x\right)\Big|_0^1 = \left(\frac{1}{3} + 1\right) - 0 = \frac{4}{3}.$$ ∎

Find the arc length of $f(x) = \log(\cos(x))$ over $[0, \pi/6]$.

$$f'(x) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$



$$1 + (f'(x))^2 = 1 + \tan^2(x) = \sec^2(x)$$

we did this before
sorry that I have to quote it

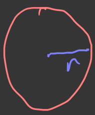
$$\ell = \int_0^{\pi/6} \sqrt{1 + [f'(x)]^2}\, dx = \int_0^{\pi/6} \sec(x)\, dx \overset{!}{=} \log|\sec(x) + \tan(x)|\Big|_0^{\pi/6}$$

$$= \log\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \log|1 + 0|$$

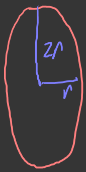
$$= \log\left|\frac{3}{\sqrt{3}}\right| - 0 = \log(\sqrt{3}).$$ ∎

There is a bit of slight of hand going on.

$C=2\pi r$



scale this by 2 in the y-axis



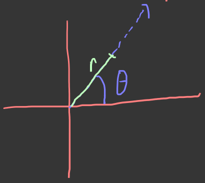
$C \neq 4\pi r$

There is no elementary relationship whatsoever between these two lengths!

There is no formula (in terms of Calc A functions) for the circumference of an ellipse.

Polar co-ordinates

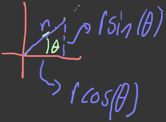
Given a pair (r, θ) we can consider taking the ray at angle θ (to the +'ve x-axis) and moving r units along it (from the origin).

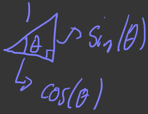


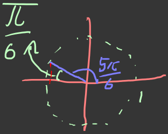
This scheme is a different co-ordinate system for reffering to points in the plane.

↗ this is the usual system

Polar co-ordinates (r, θ) $\Rightarrow$ Cartesian co-ordinates $(r\cos(\theta), r\sin(\theta))$

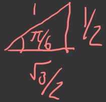






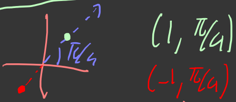
e.g. polar ⇒ cart of $\left(2, \frac{5}{6}\pi\right)$

$$\cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = \frac{-\sqrt{3}}{2}$$



$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$(r\cos\theta, r\sin\theta)$ becomes $\boxed{(-\sqrt{3}, 1)}$

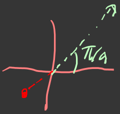


Problems with polar co-ordinates:

$(0, \theta)$ always denotes the origin $((0,0))$

(r, θ) and $(r, \theta + 2\pi)$ denote the same point.

If $r < 0$, then we go in the opposite direction



$(-1, \pi/4)$

also $(1, \pi/4 - \pi)$

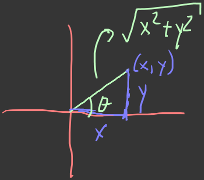
(r, θ) and $(-r, \theta + \pi)$ denote the same point or $(1, \pi/4+\pi)$

The polar representations are far from unique!

Cartesian to Polar

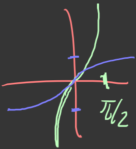
Cartesian $\qquad$ Polar

$$(x,y) \Rightarrow (?,?)$$



The distance is $\sqrt{x^2+y^2}$

$$\tan(\theta) = \frac{y}{x}$$



Consider the quantity $\arctan\left(\frac{y}{x}\right)$

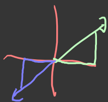
this is not defined when $x=0$

the range of arctan is $(-\pi/2, \pi/2)$

If $x>0$ then an angle in this range works

$$\tan(\theta)=\tan(\theta+\pi)$$

$$\overset{\text{Cart}}{(x,y)} \Rightarrow \overset{\text{Polar}}{\left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right)}$$



If $x<0$ then instead

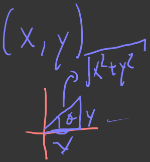
$$(x,y) \Rightarrow \left(-\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right)$$

or $\left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)+\pi\right)$

If $x = 0$

$$(0, y) \Rightarrow \left(y, \frac{\pi}{2}\right)$$

This is not maximally elegant.



$$\cos(\theta) = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin(\theta) = \frac{y}{\sqrt{x^2+y^2}}$$

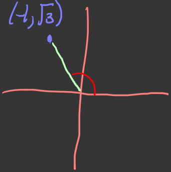
These are always defined. (Except at the origin!)

There values define an unique angle modulo 2π.

eg. cart ⇒ polar of $(-1, \sqrt{3})$

$\arctan(1) = \pi/4$

$\arctan(\sqrt{3}) = \pi/3$



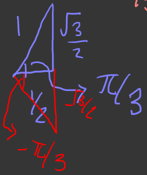
$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$$

$x < 0$

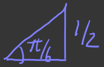
$$\theta = \arctan\left(\frac{\sqrt{3}}{-1}\right) + \pi$$

$$= \arctan(-\sqrt{3}) + \pi$$

$$= \frac{-\pi}{3} + \pi = \frac{2\pi}{3}$$



$\arcsin(1/2) = \pi/6$



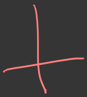
$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Cart

Polar

$(-1, \sqrt{3}) \Rightarrow (2, \frac{2\pi}{3})$.

Graphing in polar co-ordinates



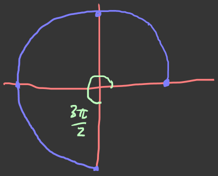
In Cartesian co-ordinates a function typically is $y = f(x)$.

In polar co-ordinates we write $r = f(\theta)$.

This gives the radius at a given angle.

$r = 1$ for $0 \leq \theta \leq \frac{3\pi}{2}$

I am plotting over the given range of θ

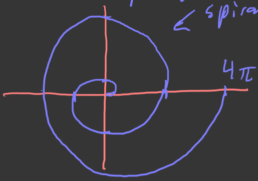


$\theta = 0$, $r = 1$

$\theta = \frac{\pi}{2}$, $r = 1$

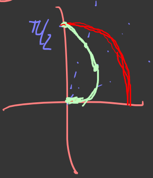
$r = \theta$ for $0 \leq \pi \leq 4\pi$

poorly drawn spiral



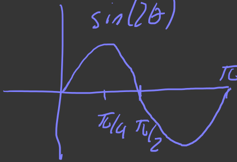
θ	r
0	0
$\pi/2$	$\pi/2$
π	π
2π	2π
4π	4π

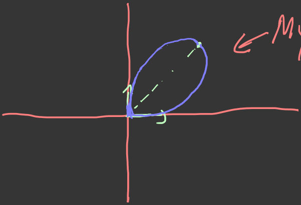
Zoom in



If $f(\alpha) = 0$ but $f(\theta) \neq 0$ near α, then the curve $r = f(\theta)$ is tangent to the line $\theta = \alpha$ at the origin.

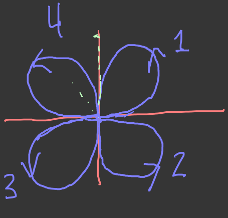
$r = \sin(2\theta)$ for $0 \leq \theta \leq \pi/2$





← My best attempt to plot this

What I continued to plot it over $0 \leq \theta \leq 2\pi$



Changing co-ordinate systems and equations

$(r, \theta) \Rightarrow (r\cos\theta, r\sin\theta)$

Cart ⇒ polar

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ $''$

(x, y)

e.g. $(x^2 + y^2)^2 = 4xy$

$2\cos(\theta)\sin(\theta) = \sin(2\theta)$

$$\Rightarrow \left(r^2\cos^2(\theta) + r^2\sin^2(\theta)\right)^2 = 4r^2\cos(\theta)\sin(\theta)$$

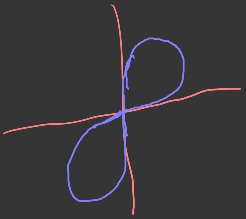
$$\Rightarrow r^4\left(\cos^2(\theta) + \sin^2(\theta)\right)^2 = 4r^2\cos(\theta)\sin(\theta)$$

$$\Rightarrow r^4 = 2r^2\sin(2\theta)$$

$$\Rightarrow r^2 = 2\sin(2\theta)$$

This has a solution only when $\sin(2\theta) \geq 0$.

For $0 \leq \theta \leq \frac{\pi}{2}$ and $\pi \leq \theta \leq \frac{3\pi}{2}$

I get $r = \sqrt{2\sin(2\theta)}$