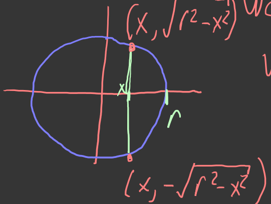
HW 3 is due at midnight tomorrow.

Area problems

What is the area of a circle?

What is the area of the region bound by the curve $x^2+y^2=r^2$. (This is a circle of radius r.)



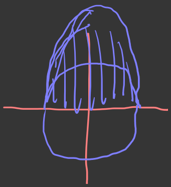
We will find the area by integrating vertical slices.

The height of the vertical slice over x is

$c^2+s^2=1$

$$h(x)=2\sqrt{r^2-x^2}$$

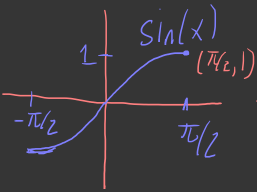
a^2-x^2

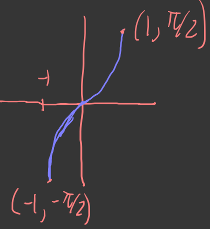


$$A=\int_{-r}^{r} h(x)\,dx = 2\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

$u=\arcsin\left(\frac{x}{r}\right)$

$$\begin{bmatrix} \text{Let } x = r\sin(u) \\ dx = r\cos(u)\,du \end{bmatrix} = 2\int_{\arcsin(-1)}^{\arcsin(1)}\sqrt{r^2-r^2\sin^2(u)}\;r\cos(u)\,du$$





$$= 2r^2\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2(u)}\,\cos(u)\,du$$

$$= 2r^2\int_{-\pi/2}^{\pi/2}\cos^2(u)\,du$$

$$2\cos^2(x) - 1 = \cos(2x)$$

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

$$= r^2\int_{-\pi/2}^{\pi/2}[1+\cos(2u)]\,du$$

$$\begin{bmatrix}\text{let } w = 2u\\ dw = 2du\end{bmatrix} = \frac{1}{2}r^2\int_{-\pi}^{\pi}[1+\cos(w)]\,dw$$

$$= \frac{1}{2}r^2(w+\sin(w))\Big|_{-\pi}^{\pi} = \frac{1}{2}r^2\big((\pi+0)-(-\pi+0)\big)$$

$$= \boxed{\pi r^2}$$

▨

Recap: $\int f(g(x))\,g'(x)\,dx \quad \begin{bmatrix}\text{let } u = g(x)\\ du = g'(x)dx\end{bmatrix}$

$$= \int f(u)\,du = F(u) + C = F(g(x)) + C$$

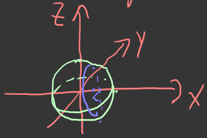
$$F(x) = \int f(x)\,dx$$

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big|_a^b$$

$$= F(u) \Big|_{g(a)}^{g(b)}$$

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Volume problems:



If we have some region in 3-space and we slice it up into x-slices (with x ranging from a to b) and each slice has area $A(x)$, then the volume of the region is

$$V = \int_a^b A(x)\, dx.$$

$$x^2 + y^2 = r^2$$

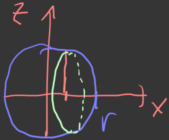
eg. What is the volume of a sphere of radius r.

This is the region bound by $x^2+y^2+z^2=r^2$.

If we fix x to be a certain value then the equation becomes $y^2+z^2 = r^2 - x^2 = R^2(x)$

the slice at x is a circle of radius $R(x) = \sqrt{r^2 - x^2}$

Thus $A(x) = \pi R^2(x) = \pi(r^2 - x^2)$



$$V = \int_{-r}^{r} A(x)\,dx$$

$$= \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= \pi\left[r^2 x - \frac{1}{3}x^3\right]\Big|_{-r}^{r}$$

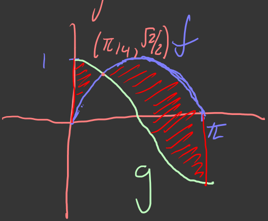
$$= \pi\left(\left(r^3 - \frac{1}{3}r^3\right) - \left(-r^3 + \frac{1}{3}r^3\right)\right)$$

$$= \pi\left(2r^3 - \frac{2}{3}r^3\right) = \pi r^3\left(\frac{6}{3} - \frac{2}{3}\right)$$

$$= \boxed{\frac{4}{3}\pi r^3.}$$

□

Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over the interval $[0, \pi]$.



We start by drawing a picture.

There is one intersection point at $x = \pi/4$.

From the graph we know

$g \geq f$ on $[0, \pi/4]$

and $f \geq g$ on $[\pi/4, \pi]$.

Thus the area is

$$A = \int_0^{\pi/4} [\cos(x) - \sin(x)]\,dx + \int_{\pi/4}^{\pi} [\sin(x) - \cos(x)]\,dx$$

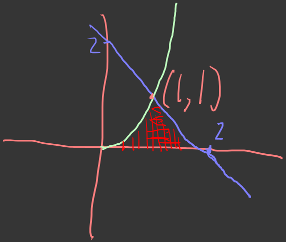
$$= \Big[\sin(x) + \cos(x)\Big]\Big|_0^{\pi/4} - \Big[\sin(x) + \cos(x)\Big]\Big|_{\pi/4}^{\pi}$$

$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right] - \left[(0-1) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right]$$

$$= \sqrt{2} - 1 + 1 + \sqrt{2} = 2\sqrt{2}. \quad \blacksquare$$

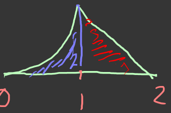
Find the area bounded by the x-axis and the curves $y = x^2$ and $y = 2 - x$.

Start by drawing a picture.



Two methods for solving this:

Method 1: Divide it into two regions



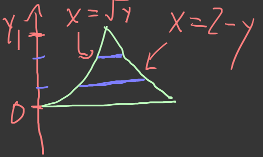
$$A = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx$$

$$= \frac{1}{3}x^3 \Big|_0^1 + \left(2x - \frac{1}{2}x^2\right)\Big|_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left((4-2) - \left(2 - \frac{1}{2}\right)\right)$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}. \quad ☒$$

Method 2: Integrate with respect to y.



$y = x^2 \qquad y = 2 - x$

So I can write $\ell(y) = 2 - y - \sqrt{y}$

$$A = \int_0^1 \ell(y)\,dy = \int_0^1 [2 - y - \sqrt{y}]\,dy$$

$$= \left[2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\right]\Big|_0^1$$

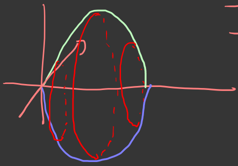
$$= \left(2 - \frac{1}{2} - \frac{2}{3}\right) - 0$$

$$= \frac{12}{6} - \frac{3}{6} - \frac{4}{6} = \frac{5}{6}. \quad ☒$$

Moral: Sometimes it is easier to integrate with respect to y.

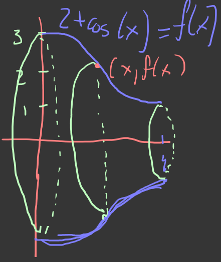
More volume problems:

Volumes of revolution:



If we rotate this region in 3-space around the x-axis (in this case) we get some 3 dim'l region. We want to find its volume.

The region between the graph of $f(x) = 2 + \cos(x)$ and the x-axis over $[0, \pi]$ is revolved around the x-axis. Find the volume of the region formed.



First we draw a picture. ✓

We determine the area of the x-slices.

$$\boxed{A(x) = \pi f^2(x)}$$

Thus we have $V = \int_0^{\pi} \pi f^2(x) dx$

$$= \pi \int_0^{\pi} (2+\cos(x))^2 dx = \pi \int_0^{\pi} (4+4\cos(x)+\cos^2(x)) dx$$

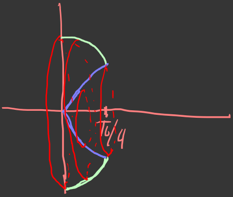
$$= \pi \int_0^{\pi} \left(4+4\cos(x)+\frac{1}{2}(1+\cos(2x))\right) dx$$

$$= \pi \left[4x+4\sin(x)+\frac{1}{2}x+\frac{1}{4}\sin(2x)\right] \Big|_0^{\pi}$$

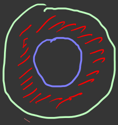
$$= \pi \left(\left(4\pi+0+\frac{1}{2}\pi+0\right)-0\right)$$

$$= \pi^2\left(4+\frac{1}{2}\right) = \frac{9}{2}\pi^2. \quad \blacksquare$$

The region between $f(x) = \cos(x)$ and $g(x) = \sin(x)$ over $[0, \pi/4]$ is revolved around the x-axis. Find the volume of the resulting region.



The x-sections look like



"washers"

anulus / anuli

Bound by two concentric circles

$$\boxed{A(x) = \pi\left(f^2(x) - g^2(x)\right)}$$

Washer method

Thus the volume is

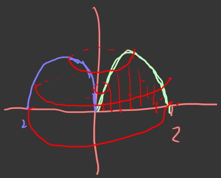
$$V = \int_0^{\pi/4} \pi\left(\cos^2(x) - \sin^2(x)\right) dx$$

$$= \pi \int_0^{\pi/4} \cos(2x)\, dx \quad \begin{bmatrix} u = 2x \\ du = 2dx \end{bmatrix}$$

$$= \frac{1}{2}\pi \int_0^{\pi/2} \cos(u)\, du$$

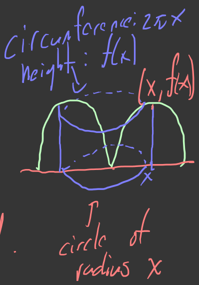
$$= \frac{1}{2}\pi \sin(u)\Big|_0^{\pi/2} = \frac{1}{2}\pi(1-0) = \frac{\pi}{2}. \square$$

The region enclosed by the x-axis and the graph of $y = 2x - x^2$ is revolved around the y-axis. Find the volume of the resulting region.



$$y = 2x - x^2$$
$$= x(2-x)$$

This is cake shaped.



Method 1: At a fixed x-value, the vertical line segment is revolved to become a "cylyndrical shell". We can compute the volume by integrating the areas of the cylyndrical shells.

$$\boxed{A(x) = 2\pi x f(x)}$$

$$A(x) = 2\pi x(2x - x^2) = 4\pi x^2 - 2\pi x^3$$

Thus the volume is

$$V = \int_0^2 A(x)\,dx = \pi\int_0^2 (4x^2 - 2x^3)\,dx$$

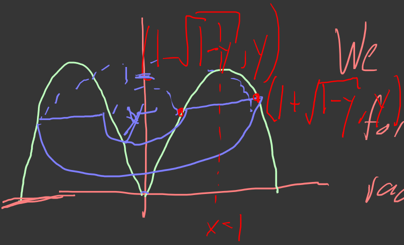
$$= \pi\left(\frac{4}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^2$$

cylindrical shell method.

$$= \pi\left(\left(\frac{32}{3} - 8\right) - 0\right)$$

$$= \pi\,\frac{32-24}{3} = \frac{8}{3}\pi. \quad \blacksquare$$

Method 2: Integrate with respect to y.



We want an expression for the outer and inner radii w.r.t. y.

$$y = -x^2 + 2x$$

at $x=1$

$y = -1+2$

$= 1$

$$y = -(x^2 - 2x)$$

$$= -((x-1)^2 - 1)$$

$$= 1 - (x-1)^2$$

$$(x-1)^2 = 1 - y$$

$$x - 1 = \pm\sqrt{1-y}$$

$$x = 1 \pm \sqrt{1-y}$$

$$A(y) = \pi\left(\left(1+\sqrt{1-y}\right)^2 - \left(1-\sqrt{1-y}\right)^2\right)$$

$$= \pi\Big(\left(1 + 2\sqrt{1-y} + (1-y)\right) - \left(1 - 2\sqrt{1-y} + (1-y)\right)\Big)$$

$$= 4\pi\sqrt{1-y}$$

$$V = \int_0 A(y)\,dy = \int_0 4\pi\sqrt{1-y}\,dy$$

$$\left[\begin{array}{l} \text{Let } u = 1-y \\ du = -dy \end{array}\right] = -\int_1^0 4\pi\sqrt{u}\,du$$

$$= \int_0^1 4\pi\sqrt{u}\,du = 4\pi\,\frac{2}{3}\,u^{3/2}\Big|_0^1$$

$$= \pi\,\frac{8}{3}. \qquad \blacksquare$$