Reminder: Midterm Monday noon - Tuesday noon.

Similar to an assignment.

$\cos(2u) = \cos^2(u) - \sin^2(u)$

$1 - \sin^2(u) = \cos^2(u)$

$1 - 2\sin^2(u)$
$= \cos(2u)$

$\sin^2(u) = \frac{1}{2}(1 - \cos(2u))$

A left-over integral from Friday:

$$\int \sqrt{\frac{x}{1-x}}\,dx \quad \left[\begin{array}{l} \text{Let } x = \sin^2(u) \\ dx = 2\sin(u)\cos(u)\,du \end{array}\right]$$

$$= \int \sqrt{\frac{\sin^2(u)}{1-\sin^2(u)}}\, 2\sin(u)\cos(u)\,du = 2\int \sqrt{\frac{\sin^2(u)}{\cos^2(u)}}\,\sin(u)\cos(u)\,du$$

$$= 2\int \sin^2(u)\,du = \int [1 - \cos(2u)]\,du = u - \frac{1}{2}\sin(2u) + C$$

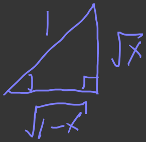
$$= u - \sin(u)\cos(u) + C = \arcsin(\sqrt{x}) - \sqrt{x}\sqrt{1-x} + C$$

$x = \sin^2(u) \qquad \sqrt{x} = \sin(u) \qquad \cos(u) = \sqrt{1 - \sin^2(u)}$

$= \sqrt{1 - (\sqrt{x})^2}$

$= \sqrt{1-x}$

$u = \arcsin(\sqrt{x})$



$$\cos(\arcsin(\sqrt{x})) = \sqrt{1-x}$$

Definite Integrals:

These are problems of the form $\int_a^b f(x)\,dx$

These can be solved by finding $F(x) = \int f(x)dx$ and computing $F(b) - F(a)$ (by FTC Part I).

You can also keep the bounds in while integrating.

1. Substitution

Suppose I have $F' = f$ and I consider

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

This implies that

$$\int_a^b f(g(x))g'(x)dx = F(g(b)) - F(g(a))$$

$F' = f$ means that $\int f(x)dx = F$

So $F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(x)dx$

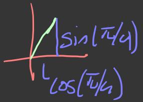
$$\int_a^b f(g(x))g'(x)dx \begin{bmatrix} \text{let } u = g(x) \\ du = g'(x)dx \end{bmatrix} = \int_{g(a)}^{g(b)} f(u)du$$

Substitution for definite integrals:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx$$

When you compute the RHS you get a number so there is no step of replacing u with x.

$$\int_0^{\pi/4} \tan(x)\sec^2(x)\,dx \quad \begin{bmatrix} \text{Let } u = \tan(x) \\ du = \sec^2(x)\,dx \end{bmatrix}$$



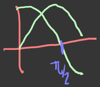
$$= \int_{\tan(0)}^{\tan(\pi/4)} u\,du = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Remember to transform the bounds.
(No step of writing in terms of x.)

$$\int_0^{\pi/6} \cos^5(3x)\,dx \quad \begin{bmatrix} \text{Let } u = 3x \\ du = 3dx \end{bmatrix}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^5(u)\,du = \frac{1}{3}\int_0^{\pi/2} (\cos^2(u))^2\cos(u)\,du$$

$$= \frac{1}{3}\int_0^{\pi/2} (1-\sin^2(u))^2 \cos(u)\, du \quad \left[\begin{array}{l} \text{Let } w = \sin(u) \\ dw = \cos(u)du \end{array}\right]$$



$$= \frac{1}{3}\int_0^{1} (1-w^2)^2\, dw = \frac{1}{3}\int_0^{1} 1-2w^2+w^4\, dw$$

$$= \frac{1}{3}\left(w - \tfrac{2}{3}w^3 + \tfrac{1}{5}w^5\right)\Big|_0^1$$

$$= \frac{1}{3}\left(\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) - 0\right)$$

$$= \frac{1}{3}\left(\frac{15}{15} - \frac{10}{15} + \frac{3}{15}\right) = \frac{8}{45}. \quad ✓$$

Parts:

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx$$

$$\int_1^e x\log(x)\,dx \quad \left[\begin{array}{ll} u = \log(x) & dv = x\,dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{array}\right]$$

$$= \frac{x^2}{2}\log(x)\Big|_1^e - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\log(x)\Big|_1^e - \left(\frac{x^2}{4}\Big|_1^e\right)$$

$$= \left(\frac{e^2}{2}1 - 0\right) - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} - \frac{1}{4} = \frac{1}{4}(e^2-1).$$

$$\left[\begin{array}{l}\log(e) = 1 \\ \log(1) = 0\end{array}\right.$$

$e^{\log(e)} = e^1 = e$

the def'n of $\log(x)$

$$\boxed{e^{\log(a)} = a}$$

$1 = e^{\log(1)} = e^0$

The Fundemental Theorem of Calculus, Part II

Recap: FTC I said: If $F' = f$ on (a,b) and F is cts. on $[a,b]$, and f is integrable then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is important because it computes definite integrals.

Recall how hard $\int_0^1 x^2\,dx$ this was with Riemann sums.

But now we can do $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$ ✓

This establishes the importance of antiderivatives

Is this a robust method?

Will an antiderivative always exist?

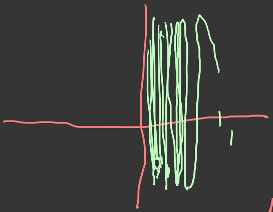
Differentiable at a $\Rightarrow$ continuous at a.

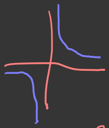
Suppose that $g = f'$. What can we say about g?

You may guess that g must be continuous.

But this is not generally true.

$\sin(1/x)$





$-\frac{2}{x^3}$

$f'(0)=0$

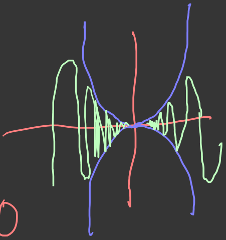
$$\boxed{f = x^2 \sin(1/x^2)}$$

$f(0)=0$

$$\lim_{h\to 0} \frac{h^2 (\sin 1/h^2)}{h} = \lim_{h\to 0} h \sin(1/h^2) = 0$$

$$f' = 2x \sin(1/x^2) + x^2 \cos(1/x^2) \frac{-2}{x^3}$$

$$= 2x\sin(1/x^2) - 2 \frac{1}{x} \cos(1/x^2)$$



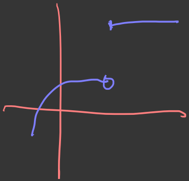
f' is defined at 0

but f' is not continuous at

Essential singularity

the function takes on all real values arbitrarily close to the discontinuity

(approximately correct)



← Jump discontinuity

Theorem: If $g = f'$, then g has no jump discontinuities.

On HW 1 we considered

$$f(x) = \begin{cases} 0 & x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

f has a jump discontinuity $\Rightarrow$ there does not exist F with $F' = f$.

This f does not have an antiderivative.

But we know that f is integrable on $[0,1]$

We showed $\int_0^1 f(x)\,dx = 1/2$.

In other words: The strategy of finding an antiderivative does not always work.

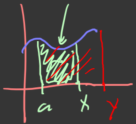
Integrable functions may lack an antiderivative.

Thm: FTC Part II says that continuous functions always have an antiderivative.

More specifically, suppose f is continuous on $[a,b]$

then for any $x \in [a,b]$ consider $F(x)$

$$F(x) = \int_a^x f(t)\,dt$$



Because f is cts. $[a,x]$, F is always defined.

Then the theorem is that $\frac{d}{dx}F(x) = f(x)$.

This is theoretically interesting, but useless for computing definite integrals.

This theorem can be written as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This is often explained as saying that differentiation and integration are inverse operations.

Super informal proof sketch:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \approx \lim_{h\to 0} \frac{1}{h}\, h f(x) = f(x).$$

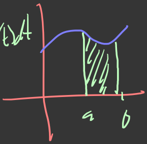
When h is small t is very close to x

And $f(t) \approx f(x)$ approximately constant.

Then $\int_x^{x+h} f(t)\,dt \approx \int_x^{x+h} f(x)\,dt = h f(x)$, $\boxed{}$

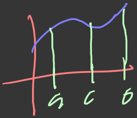
FTC Part II leads to some questions:

$\int_a^b f(t)\,dt$



First: $\int_a^a f(t)\,dt = 0$

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$



With these conventions we have

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

$$F(x) = \int_a^x \frac{1}{1+\sin^2(t)}\,dt \qquad F(5) = \int_a^5 \frac{1}{1+\sin^2(t)}\,dt$$

$$F'(x) = \frac{1}{1+\sin^2(x)}$$

Let $C(x) = x^3$

Then $f(x) = F(C(x))$

$= (F \circ C)(x)$

$= F(x^3)$

What about $f(x) = \displaystyle\int_a^{x^3} \frac{1}{1+\sin^2(t)}\,dt$

$$f'(x) = \frac{3x^2}{1+\sin^2(x^3)}$$

$$\frac{1}{1+\sin^2(x^3)} = F'(x^3) = F'(C(x))$$

$f = F \circ C \qquad f' = (F' \circ C) \cdot C'$

$$f'(x) = F'(C(x))\, C'(x)$$

What about

$g = C \circ F$, $\boxed{g(x) = \left(\int_a^x \frac{1}{1+\sin^2(t)}\,dt\right)^3}$

$$g'(x) = C'(F(x))\, F'(x)$$

$$= 3\left(\int_a^x \frac{1}{1+\sin^2(t)}\,dt\right)^2 \frac{1}{1+\sin^2(x)}$$

$\sin(x)$

x^{10}

$\left(\left(x^2+\sin(x)\right)^{10}\right)'$

$$= 10\left(x^2+\sin(x)\right)^9\left(2x+\cos(x)\right)$$

Today: I added a basic function to your Calc A toolbox, i.e. $\boxed{F(x)=\int_a^x f(t)\,dt}$

$$\boxed{h(x) = \int_a^{\int_a^{x^3} \frac{1}{1+\sin^2(t)}\,dt} \frac{1}{1+\sin^2(t)}\,dt}$$

$$(F\circ C)' = (F'\circ C)\cdot C'$$

$$h(x) = F(F(C(x)))$$

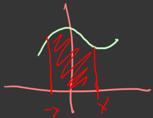
$$h'(x) = F'(F(C(x)))\,F'(C(x))\,C'(x)$$

$$(F\circ(F\circ C))' = (F'\circ(F\circ C))\cdot(F\circ C)'$$

$$= (F'\circ F\circ C)\cdot(F'\circ C)\cdot C'$$

$$= \frac{1}{1+\sin^2\left(\int_a^{x^3} \frac{1}{1+\sin^2(t)}\,dt\right)} \frac{1}{1+\sin^2(x^3)}\, 3x^2$$

Let $F(x) = \int_{-x}^{x} f(t)\,dt$



$$= \int_{-x}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

$$= -\int_{0}^{-x} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

$$F'(x) = -f(-x)(-1) + f(x) = f(-x) + f(x)$$

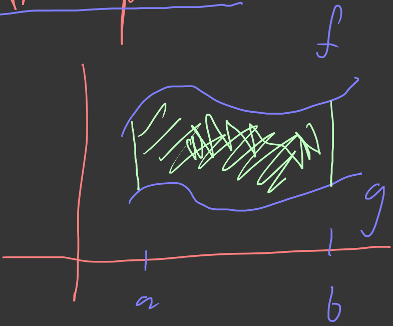
We are done with theory.

You have a complete Calc B toolbox.

The rest of the course is applications.

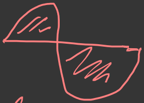
(i.e. word problems)

Area problems



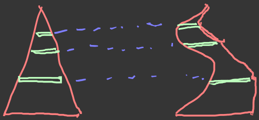
eg. compute the shaded area.

Integrals were a computation of the area under the curve.

Those regions looked like 

This region is not of that form.

Cavalieri's theorem (Historically significant)

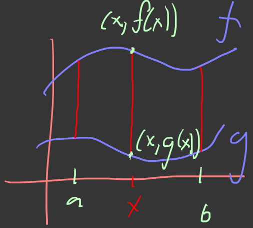


If the cross sections have the same length then the areas are equal.

We do not have a general notion of area, so we can't prove this.

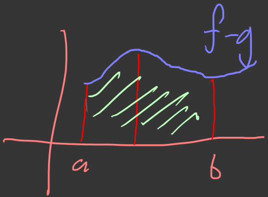
↳ eg. Lebesgue measure

We take this as an assumption.



$f \geq g$ on $[a,b]$

The height of the slab at x is $f(x) - g(x)$



$$\int_a^b [f(x)-g(x)]\,dx$$

This should give the area between the regions.

If $f \geq g$ on $[a,b]$

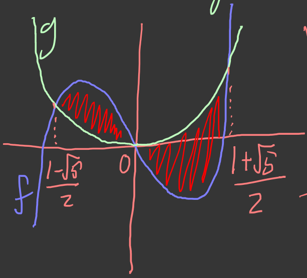
then the area between f and g over $[a,b]$

is $\int_a^b f(x) - g(x)\,dx$.

Example:

Find the area of the region bounded by the curves $f(x) = x^3 - x$ and $g(x) = x^2$.

I haven't given bounds.



We don't include the infinite unbounded areas to the sides.

I want a sum of these two areas.

First we solve for the bounds of integration:

$$x^3 - x = x^2 \Rightarrow x^3 - x^2 - x = 0$$

$$\Rightarrow x(x^2-x-1)=0$$

$$\Rightarrow x=0, \frac{1\pm\sqrt{5}}{2}$$

I have the intervals

$$\left[\frac{1-\sqrt{5}}{2}, 0\right] \text{ and } \left[0, \frac{1+\sqrt{5}}{2}\right]$$

I want to know which of f or g is bigger.

I can determine this by plugging in numbers

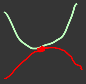
$f(x)=x^3-x$

$g(x)=x^2$

$$\frac{1-\sqrt{5}}{2} < \frac{1-\sqrt{4}}{2} = \frac{-1}{2} < 0$$

$$f(-1/2) = \frac{-1}{8}+\frac{1}{2} = \frac{-1}{8}+\frac{4}{8} = \frac{3}{8}$$

$$g(-1/2) = (-1/2)^2 = 1/4 = 2/8$$



Thus $f \geq g$ $\left[\frac{1-\sqrt{5}}{2}, 0\right]$

$$0 < 1 < \frac{1+\sqrt{5}}{2}$$

$f(1) = 1 - 1 = 0$ $\quad$ $g \geq f$ on $\left[0, \frac{1+\sqrt{5}}{2}\right]$

$g(1) = 1^2 = 1$

Thus the area is given by:

$$A = \int_{\frac{1-\sqrt{5}}{2}}^{0} f(x) - g(x)\,dx + \int_{0}^{\frac{1+\sqrt{5}}{2}} g(x) - f(x)\,dx$$

$$= \int_{\frac{1-\sqrt{5}}{2}}^{0} (x^3 - x - x^2)\,dx + \int_{0}^{\frac{1+\sqrt{5}}{2}} (x^2 - (x^3 - x))\,dx$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{\frac{1-\sqrt{5}}{2}}^{0} + \left(\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^2}{2}\right)\Bigg|_{0}^{\frac{1+\sqrt{5}}{2}}$$

Now you evaluate this.

In this example it's not very nice.