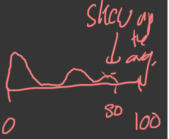
The Midterm will be a 24h take-home exam.

12:00 Monday – 12:00 Tuesday (noon) in 6days
14 June 15 June

Open book. No online sources (yes books,
No Chegg, no looking up questions, yes
lecture videos, yes assignment solutions).

HW 1 average (in progress) ~40%
Recall this will be curved a lot.



Material: Up to FTC Part II
Including today's lecture, + a bit from Thursday.

Similar to the assignment (just a bit shorter).
~2/3 of an assignment.

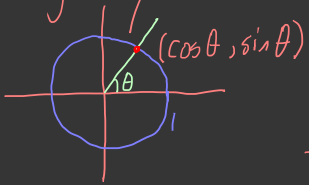
Questions are similar to previous assignments.
In other words your past experiance makes these questions quiker to solve.

Best way to study: Go over the assignments + the posted solutions. Look at HW 3. (Due Wed @ 11:59pm)

office hours

Ending point of the material will be questions of the form, let $F(x) = \int_0^{x^2\sin(x)} \cos^2(t)\,dt$ find $F'(x)$. (We'll do this at the start of Thursday's lecture.) There will be a Riemann sums question. (Look at the solutions.)

Trig Day (wow, so much trig)



The unit circle is described by the equation $x^2+y^2=1$.

Therefore we get the Pythagorean thm:

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Dividing by $\cos^2(\theta)$ yields

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

(Pythagoras V2)

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$$

Special cases: $\alpha = \beta = \theta$

$$\underline{\cos(2\theta)} = \cos^2(\theta) - \sin^2(\theta)$$

$$= \cos^2(\theta) - (1-\cos^2(\theta))$$

$$= 2\cos^2(\theta) - 1$$

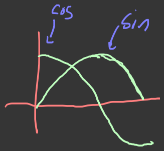
or $$= (1-\sin^2(\theta)) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$$

$$\underline{\sin(2\theta)} = 2\sin(\theta)\cos(\theta)$$

Integrating using trig identities:

We know $2\cos^2(\theta) = \cos(2\theta)+1$

$$\int\sqrt{1+\sin(2x)}\,dx = \int\sqrt{1+\cos(2x-\pi/2)}\,dx = (*)$$



$$\sin(x) = \cos(x-\pi/2)$$

$$(*) = \int\sqrt{1+\cos(2(x-\pi/4))}\,dx$$

$$= \int\sqrt{2\cos^2(x-\pi/4)}\,dx$$

$$= \sqrt{2}\int\cos(x-\pi/4)\,dx$$

$$= \boxed{\sqrt{2}\sin(x - \pi/4) + C}$$

Method 2:

$(\cos(x)+\sin(x))^2 = \cos^2(x) + 2\sin(x)\cos(x) + \sin^2(x)$

$$\int \sqrt{1+\sin(2x)}\,dx = \int \sqrt{1+2\sin(x)\cos(x)}\,dx$$

$$= \int \sqrt{\cos^2(x)+\sin^2(x)+2\sin(x)\cos(x)}\,dx$$

$1 = \cos^2(x)+\sin^2(x)$

$$= \int \sqrt{(\cos(x)+\sin(x))^2}\,dx = \int \cos(x)+\sin(x)\,dx$$

$$= \boxed{\sin(x) - \cos(x) + C}$$

HW 1 Q8 shows why these two are equal

$$\int \cos^2(x)\,dx = \int \frac{1}{2}(\cos(2x)+1)\,dx$$

$\left[\begin{array}{l} \text{let } u = 2x \\ du = 2dx \end{array}\right]$

$\hookrightarrow dx = \frac{1}{2}du$

$$= \frac{1}{4}\int (\cos(u)+1)\,du = \frac{1}{4}(\sin(u)+u)+C$$

$$= \frac{1}{4}(\sin(2x)+2x)+C = \frac{1}{4}\sin(2x) + \frac{1}{2}x + C$$

$$= \frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x + C$$

This trick works for $\int \cos^{2n}(x)dx$ and $\int \sin^{2n}(x)dx$

first you do

$\sin^2(x) = 1-\cos^2(x)$

$$\begin{aligned}\cos(2x) &= \cos^2(x)-\sin^2(x)\\ &= \cos^2(x)-(1-\cos^2(x))\\ &= 2\cos^2(x)-1\end{aligned}$$

$$\Rightarrow 2\cos^2(x) = \cos(2x)+1$$

$$\Rightarrow \cos^2(x) = \frac{1}{2}(\cos(2x)+1)$$

Lesson 2 of trig day:

The "300 IQ" tan half angle substitution

Suppose we set $u = \tan\left(\frac{x}{2}\right)$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}$$

$$= \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)} = \frac{2u}{1-u^2}$$

$$(1+x)^2 = 1+2x+x^2$$

$$\sin^2(x)+\cos^2(x)=1 \Rightarrow \tan^2(x)+1 = \sec^2(x)$$

$$(1+x^2)^2 = 1+2x^2+x^4$$

$$\sec^2(x) = 1 + \underbrace{\left(\frac{2u}{1-u^2}\right)}_{}{}^2 = \frac{(1-u^2)^2+4u^2}{(1-u^2)^2}$$

$$\tan(x)$$

$$= \frac{1 - 2u^2 + u^4 + 4u^2}{(1-u^2)^2} = \frac{1 + 2u^2 + u^4}{(1-u^2)^2} = \frac{(1+u^2)^2}{(1-u^2)^2}$$

$$\sec(x) = \frac{1+u^2}{1-u^2} \qquad \sec(x) = \frac{1}{\cos(x)}$$

$$\cos(x) = \frac{1-u^2}{1+u^2} \qquad \sin(x) = \tan(x)\cos(x)$$

$$= \frac{2u}{1-u^2} \, \frac{1-u^2}{1+u^2} = \frac{2u}{1+u^2}$$

$$u = \tan\left(\frac{x}{2}\right)$$

$$x = 2\arctan(u) \qquad dx = \frac{2}{1+u^2}\,du$$

In summary, if we set $u = \tan\left(\frac{x}{2}\right)$

then $\cos(x) = \frac{1-u^2}{1+u^2}$, $\sin(x) = \frac{2u}{1+u^2}$

$$dx = \frac{2}{1+u^2}\,du$$

If we have some rational expression of sin and cos, this turns the problem into a rational function.

$$\int \csc(x)\,dx = \int \frac{dx}{\sin(x)} \quad \left[\text{Let } u = \tan\left(\frac{x}{2}\right)\right]$$

$$= \int \frac{1}{\left(\frac{2u}{1+u^2}\right)} \, \frac{2}{1+u^2}\,du = \int \frac{du}{u} = \log|u| + C$$

$$= \log\left|\tan\left(\frac{x}{2}\right)\right| + C$$

$$\int \frac{dx}{4+5\cos(x)} \quad \left[\text{Let } u = \tan\left(\frac{x}{2}\right)\right] = \int \frac{1}{4+5\left(\frac{1-u^2}{1+u^2}\right)} \, \frac{2}{1+u^2}\,du$$

$$= \int \frac{2\,du}{4(1+u^2)+5(1-u^2)} = \int \frac{2\,du}{9-u^2} = -\int \frac{2\,du}{u^2-9} = (*)$$

$$\frac{2}{u^2-9} = \frac{2}{(u+3)(u-3)} = \frac{A}{u+3} + \frac{B}{u-3}$$

$$= \frac{A(u-3) + B(u+3)}{(u+3)(u-3)} = \frac{(A+B)u + (3B-3A)}{(u+3)(u-3)}$$

$$\Rightarrow \quad A+B=0 \qquad \Rightarrow \quad -A=B$$

$$3B-3A=2 \qquad 3B-3A = 3B+3B = 6B$$

$$\Rightarrow \quad 6B=2 \quad \Rightarrow \quad B=\frac{1}{3}$$

$$\Rightarrow \frac{2}{u^2-9} = \frac{1}{3}\left[\frac{1}{u-3} - \frac{1}{u+3}\right] \qquad \Rightarrow A = \frac{-1}{3}$$

$$(*) = \frac{1}{3}\int \left[\frac{1}{u+3} - \frac{1}{u-3}\right] du = \frac{1}{3}\left[\ln|u+3| - \ln|u-3|\right] + C$$

$$= \frac{1}{3}\left[\ln\left|\tan\left(\frac{x}{2}\right)+3\right| - \ln\left|\tan\left(\frac{x}{2}\right)-3\right|\right] + C$$

Interlude: What if none of the tricks work?

Consider $\int \sqrt{\sin(x)}\,dx$

this integral cannot be expressed in terms of the functions that we know.

While the function is integrable the problem of "finding a formula is impossible".

It's very hard to tell if an integral is even possible. (in terms of Calc A functions)

We introduce $Si(x) = \int_0^x \sqrt{\sin(t)}\,dt$

this is a new function (some calculators have a button for this).

i.e. yes integration is hard!!

no direct algorithm, just a bunch of tricks

Lesson 3 of trig day: Reverse substitutions

$u = \arcsin(x)$

$$\int \sqrt{1-x^2}\,dx \quad \begin{bmatrix} \text{Let } x = \sin(u) \\ dx = \cos(u)\,du \end{bmatrix}$$

$\cos^2(\theta) =$
$\frac{1}{2}(\cos(2\theta)+1)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
$= 2\cos^2(\theta) - 1$

$$= \int \sqrt{1-\sin^2(u)}\,\cos(u)\,du$$

$$= \int \sqrt{\cos 2(u)}\,\cos(u)\,du$$

$$= \int \cos^2(u)\,du = \int \frac{1}{2}(\cos(2u)+1)\,du = \dots$$

$$= \frac{1}{2}\sin(u)\cos(u) + \frac{1}{2}u + C$$ (from before)

$$= \frac{1}{2}\sin(\arcsin(x))\cos(\arcsin(x)) + \frac{1}{2}\arcsin(x) + C$$

You are not allowed to have answers with trig functions of inverse trig functions, i.e. this is not simplified

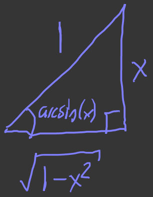
$$= \frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$$

$\sin = \frac{O}{H}$

$\cos = \frac{A}{H}$

$$\cos(\arcsin(x))$$

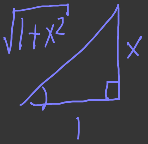
$$= \sqrt{1-x^2}$$



$\tan = \frac{O}{A} = x$

$$\sin(\arctan(x))$$

$$= \frac{x}{\sqrt{1+x^2}}$$

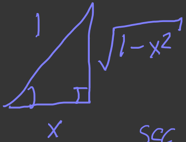


$\sin = \frac{O}{H}$

$\cos = \frac{A}{H} = x$

$$\sec(\arccos(x))$$

$$= 1/x$$



$\sec = \frac{1}{\cos}$

1 H

$$= \frac{}{(A/H)} = \frac{}{A}$$

$$\csc(\arccos(x))$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\csc = \frac{1}{\sin} = \frac{1}{(O/H)} = \frac{H}{O}$$

General idea: use pythagoras to write something as a square

$$\sin^2(x) + \cos^2(x) = 1 \Rightarrow \boxed{1 - \sin^2(x) = \cos^2(x)}$$

$$\boxed{\tan^2(x) + 1 = \sec^2(x)} \Rightarrow \boxed{\sec^2(x) - 1 = \tan^2(x)}$$

If I have	I substitute
$a^2 + x^2$	$x = a\tan(u)$
$a^2 - x^2$	$x = a\sin(u)$
$x^2 - a^2$	$x = a\sec(u)$

$\sqrt{x^2-a^2}$ vs. $\sqrt{a^2-x^2}$

$$\sqrt{a^2(1+x^2)}$$
$$= \sqrt{a^2}\sqrt{1+x^2}$$
$$= a\sqrt{1+x^2}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} \quad \left[\begin{array}{l} \text{let } x = a\tan(u) \\ dx = a\sec^2(u)\,du \end{array}\right]$$

$$= \int \frac{a\sec^2(u)\,du}{\sqrt{a^2+a^2\tan^2(u)}} = \int \frac{\sec^2(u)\,du}{\sqrt{1+\tan^2(u)}}$$

$$= \int \frac{\sec^2(u)\,du}{\sqrt{\sec^2(u)}} = \int \sec(u)\,du$$

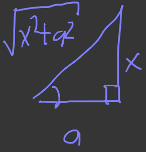
$$u = \arctan\left(\frac{x}{a}\right)$$

(from last time)
$$= \ln\left|\sec(u)+\tan(u)\right| + C$$

$$= \ln\left|\sec\left(\arctan\left(\frac{x}{a}\right)\right) + \tan\left(\arctan\left(\frac{x}{a}\right)\right)\right| + C$$

↑ this is unacceptable

$$= \ln\left|\sec\left(\arctan\left(\frac{x}{a}\right)\right) + \frac{x}{a}\right| + C$$



$$\tan = \frac{O}{A} = \frac{x}{a}$$

$$\sec = \frac{1}{\cos} = \frac{1}{(A/H)} = \frac{H}{A}$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C$$

$$= \ln\left|\sqrt{\left(\frac{x}{a}\right)^2+1} + \frac{x}{a}\right| + C$$

↑ this is acceptable

$u = \operatorname{arcsec}\left(\frac{x}{3}\right)$

$$\int \frac{dx}{(x^2-9)^{3/2}} \quad \left[\begin{aligned} &\text{Let } x = 3\sec(u) \\ &dx = 3\sec(u)\tan(u)\,du \end{aligned}\right]$$

$$= \int \frac{3\sec(u)\tan(u)\,du}{(9\sec^2(u)-9)^{3/2}} = \int \frac{3\sec(u)\tan(u)\,du}{(3^2)^{3/2}(\sec^2(u)-1)^{3/2}}$$

$$= \int \frac{3}{27} \frac{\sec(u)\tan(u)}{(\sec^2(u)-1)^{3/2}}\,du$$

$$(x^2)^{3/2} = \left((x^2)^{1/2}\right)^3 = (x)^3 = x^3$$

$$(x^a)^b = x^{ab}$$

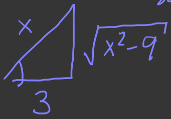
$$= \frac{1}{9}\int \frac{\sec(u)\tan(u)}{(\tan^2(u))^{3/2}}\,du = \frac{1}{9}\int \frac{\sec(u)\tan(u)\,du}{\tan^3(u)}$$

$$= \frac{1}{9}\int \frac{\sec(u)}{\tan^2(u)}\,du = \frac{1}{9}\int \frac{1}{\cos(u)}\,\frac{\cos^2(u)}{\sin^2(u)}\,du$$

$$= \frac{1}{9}\int \frac{1}{\sin(u)}\,\frac{\cos(u)}{\sin(u)}\,du = \frac{1}{9}\int \csc(u)\cot(u)\,du$$

$$= -\frac{1}{9}\csc(u) + C = -\frac{1}{9}\csc\left(\operatorname{arcsec}\left(\frac{x}{3}\right)\right) + C$$

$$\sec = \frac{H}{A} = \frac{x}{3}$$



$$\csc'(x) = -\csc(x)\cot(x)$$

$$\csc = \frac{H}{O}$$

$$= -\frac{1}{9}\,\frac{x}{\sqrt{x^2-9}} + C$$

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx \quad \left[\begin{array}{l} \text{Let } x = \sin(u) \\ dx = \cos(u)\,du \end{array}\right] \quad u = \arcsin(x)$$

$$= \int \frac{\arcsin(\sin(u))\cos(u)\,du}{\sqrt{1-\sin^2(u)}}$$

$$= \int \frac{u\cos(u)}{\sqrt{\cos^2(u)}}\,du = \int u\,du = \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\arcsin^2(x) + C.$$

$$\int (4-3x^2)^{-5/2}\,dx$$

$u = \arcsin\left(\frac{\sqrt{3}}{2}x\right)$

$$= \int \left(3\left(\frac{4}{3}-x^2\right)\right)^{-5/2} dx$$

$$\left[\begin{array}{l} \text{let } x = \frac{2}{\sqrt{3}}\sin(u) \\ dx = \frac{2}{\sqrt{3}}\cos(u) \end{array}\right]$$

$$= \int \left(3\,\frac{4}{3}\left(1-\sin^2(u)\right)\right)^{-5/2} \frac{2}{\sqrt{3}}\cos(u)\,du$$

$$= \frac{2}{\sqrt{3}}\int \left(2^2\cos^2(u)\right)^{-5/2}\cos(u)\,du$$

$$\left(\left(2\cos(u)\right)^2\right)^{-5/2}$$

$$= \left(2\cos(u)\right)^{-5}$$

$$= \frac{1}{\left(2\cos(u)\right)^5}$$

$$= \frac{2}{\sqrt{3}}\int \frac{\cos(u)}{2^5\cos^5(u)}\,du$$

$$= \frac{1}{16\sqrt{3}}\int \frac{1}{\cos^4(u)}\,du = \frac{1}{16\sqrt{3}}\int \sec^4(u)\,du$$

$$= \frac{1}{16\sqrt{3}}\int \sec^2(u)\left(\tan^2(u)+1\right)du$$

$$= \frac{1}{16\sqrt{3}}\left[\int \sec^2(u)\,du + \int \sec^2(u)\tan^2(u)\,du\right]$$

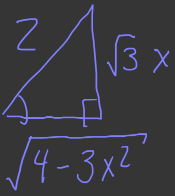
$$= (*)$$

$$\int \sec^2(u)\tan^2(u)\,du \quad \left[\begin{array}{l} \text{Let } w = \tan(u) \\ dw = \sec^2(u)\,du \end{array}\right]$$

$$= \int w^2\,dw = \frac{w^3}{3} + C = \frac{1}{3}\tan^3(u) + C$$

$$(*) = \frac{1}{16\sqrt{3}}\left[\tan(u) + \frac{1}{3}\tan^3(u)\right] + C$$

$$= \frac{1}{16\sqrt{3}}\left[\tan\left(\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right) + \frac{1}{3}\tan^3\left(\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right)\right] + C$$



$$\sin = \frac{O}{H} = \frac{\sqrt{3}\,x}{2}$$

$$\tan = \frac{O}{A}$$

$$\tan\left(\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right) = \frac{\sqrt{3}\,x}{\sqrt{4-3x^2}}$$

$$= \frac{1}{16\sqrt{3}}\left[\frac{\sqrt{3}\,x}{\sqrt{4-3x^2}} + \frac{1}{3}\,\frac{3\sqrt{3}\,x^3}{(4-3x^2)^{3/2}}\right] + C$$

$$= \frac{1}{16}\left[\frac{x}{\sqrt{4-3x^2}} + \frac{x^3}{(4-3x^2)^{3/2}}\right] + C$$