Calculus B, Lecture 4

HW 2 has been posted and is due Wed. @ 11:59 pm.

$$\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx = \int \frac{\cos(x)}{\cos^2(x)}\,dx = \int \frac{\cos(x)}{1-\sin^2(x)}\,dx$$

$$\begin{bmatrix} \text{Let } u = \sin(x) \\ du = \cos(x)\,dx \end{bmatrix} = \int \frac{du}{1-u^2} = -\int \frac{du}{u^2-1} = \frac{1}{2}\int \left[\frac{1}{u+1} - \frac{1}{u-1}\right] du$$

$$\frac{1}{u-1} - \frac{1}{u+1} = \frac{u+1-(u-1)}{u^2-1} = \frac{2}{u^2-1}$$

$$\frac{1}{2}\int \left[\frac{1}{u+1} - \frac{1}{u-1}\right] du = \frac{1}{2}\Big[\ln|u+1| - \ln|u-1|\Big] + C$$

$$= \frac{1}{2}\ln\left|\frac{\sin(x)+1}{\sin(x)-1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(\sin(x)+1)^2}{\sin^2 x - 1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sin(x)+1}{\cos(x)}\right|^2 + C$$

$$= \ln|\tan(x) + \sec(x)| + C$$

3. Integration by parts

$(x^2)' = 2x$

$x' = 1$

$x' x' = 1$

Product rule:

$$(fg)' = f'g + fg'$$

Integrate this

$$\int (f \cdot g)'(x)\,dx = \int [(f' \cdot g)(x) + (f \cdot g')(x)]\,dx$$

$$\Rightarrow (f \cdot g)(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\Rightarrow \boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

Integration by parts.

$u = f(x) \qquad du = f'(x)\,dx$

$v = g(x) \qquad dv = g'(x)\,dx$

Short form

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\int x e^x\,dx \quad \begin{bmatrix} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{bmatrix}$$

$$= xe^x - \int e^x dx = xe^x - e^x + C$$

$$\int x \sin(x) dx \quad \begin{bmatrix} u = x & dv = \sin(x) dx \\ du = dx & v = -\cos(x) \end{bmatrix}$$

$$= -x\cos(x) + \int \cos(x) dx = -x\cos(x) + \sin(x) + C$$

$$\int \log(x) dx \quad \begin{bmatrix} u = \log(x) & dv = 1 \cdot dx \\ du = 1/x \, dx & v = x \end{bmatrix}$$

$$= x\log(x) - \int x \, 1/x \, dx = x\log(x) - x + C$$

Moral: sometimes we integrate the implicit 1.

$$\int \frac{\log(x)}{x} dx \quad \begin{bmatrix} u = \log(x) & dv = 1/x \, dx \\ du = 1/x \, dx & v = \log(x) \end{bmatrix}$$

$$= \log^2(x) - \int \frac{\log(x)}{x} dx$$

$$\Rightarrow 2\int \frac{\log(x)}{x} dx = \log^2(x)$$

$$\Rightarrow \int \frac{\log(x)}{x} dx = \frac{1}{2}\log^2(x) + C$$

Remember to add the +C at the end

Moral: sometimes you get back to the starting point.

$$\int e^x \sin(x)\,dx \quad \begin{bmatrix} u = e^x & dv = \sin x\,dx \\ du = e^x dx & v = -\cos(x) \end{bmatrix}$$

$$= -e^x\cos(x) + \int e^x \cos(x)\,dx \quad \begin{bmatrix} u = e^x & dv = \cos(x)dx \\ du = e^x dx & v = \sin(x) \end{bmatrix}$$

$$= -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x)\,dx$$

$$\Rightarrow 2\int e^x \sin(x)\,dx = e^x(\sin(x) - \cos(x))$$

$$\Rightarrow \int e^x \sin(x)\,dx = \frac{1}{2}e^x(\sin(x) - \cos(x)) + C.$$

Moral: We often do parts multiple times.

$$\int (x^2+1)\sin(x)\,dx \quad \begin{bmatrix} u = x^2+1 & dv = \sin(x)dx \\ du = 2x\,dx & v = -\cos(x) \end{bmatrix}$$

$$= -(x^2+1)\cos(x) + 2\int x\cos(x)\,dx \quad \begin{bmatrix} u = x & dv = \cos(x)dx \\ du = dx & v = \sin(x) \end{bmatrix}$$

$$= -(x^2+1)\cos(x) + 2\left[x\sin(x) - \int \sin(x)\,dx\right]$$

$= -(x^2+1)\cos(x) + 2x\sin(x) + 2\cos(x) + C$

$\int \arctan(x)\,dx$

parts

$$\left[\begin{array}{ll} u = \arctan(x) & dv = 1\cdot dx \\ du = \dfrac{dx}{1+x^2} & v = x \end{array}\right]$$

$= x\arctan(x) - \int \dfrac{x}{1+x^2}\,dx$

substitution

$$\left[\begin{array}{l} \text{Let } u = x^2+1 \\ du = 2x\,dx \end{array}\right]$$

$\frac{1}{2}du = x\,dx$

$= x\arctan(x) - \frac{1}{2}\int \dfrac{du}{u}$

$= x\arctan(x) - \frac{1}{2}\log|u| + C$

$= x\arctan(x) - \frac{1}{2}\log|x^2+1| + C$

Moral: We often apply both parts and substitution.

4. Partial fraction (algebra heavy)

$ax^2+bx+c = 0 \qquad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\sqrt{b^2-4ac}$ exists only when $b^2-4ac \geq 0$

if $b^2-4ac < 0$

then the quadratic is irreducible

For quadratics of the form x^2+bx+c
the equation has real roots when $b^2 \geqslant 4c$

If $b^2 < 4c$, then it's irreducible. b^2-4c is the discriminant

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2+3/4}$$

completing the square

$$\frac{4}{3} = \left(\frac{2}{\sqrt{3}}\right)^2$$

$$= \frac{4}{3}\int \frac{dx}{\underbrace{\left(\frac{2}{\sqrt{3}}(x+1/2)\right)^2}_{}+1}$$

$$\left[\begin{array}{l} \text{Let } u = \frac{2}{\sqrt{3}}(x+1/2) \\ du = \frac{2}{\sqrt{3}}\,dx \end{array}\right]$$

$$= \frac{2}{\sqrt{3}}\int \frac{du}{u^2+1}$$

$$= \frac{2}{\sqrt{3}}\arctan(u) + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$

$$x^2+bx+c = \underbrace{(x+b/2)^2}_{} + \underbrace{(c-b^2/4)}_{}$$

if $b^2 < 4c$ then
$c - b^2/4 > 0$
then I've written
x^2+bx+c
as a square + p've number

$$(x+b/2)^2 = x^2+bx+b^2/4$$

This is a general reccpie for $\int \frac{dx}{x^2+bx+c}$
when $b^2<4c$.

Def: A rational function is a quotient of two polynomials.

eg. $\frac{x}{x^2-1}$, $\frac{x^{10}+x^4+2}{x^2-7}$, $\frac{1}{x^3-3x^2+x-5}$

$\frac{x^2+1}{1}$, $\frac{0}{1}$, $\frac{1}{1}$

The partial fractions technique lets you integrate ANY rational function whatsoever,
i.e. we use partial fractions when we see rational functions.

THE METHOD:

Step 1: If $\frac{f}{g}$ is a rational function (f,g are polynomials) then we have

$\frac{f}{g} = q + \frac{r}{g}$ where q and r are poly. and $\deg(r) < \deg(g)$.

i.e. polynonomial long division exists

$$\frac{x^3+x+1}{x^2-2x-3} \quad \begin{matrix} \rightarrow \text{degree } 3 \\ \rightarrow \text{degree } 2 \end{matrix}$$

Polynomial long division

$$\begin{array}{r|l} & x+2 \\ \hline x^2-2x-3 & x^3+0x^2+x+1 \\ & \underline{x^3-2x^2-3x} \\ & \quad 2x^2+4x+1 \\ & \quad \underline{2x^2-4x-6} \\ & \qquad 8x+7 \end{array}$$

$$\frac{x^3+x+1}{x^2-2x-3} = x+2+\frac{\overset{\text{degree } 1}{8x+7}}{x^2-2x-3}$$

$\frac{f}{g} = q + \frac{r}{g}$ with $\deg(r) < \deg(g)$

If I have $\frac{f}{g}$ with $\deg(f) \geq \deg(g)$

then I start by dividing.

Example:
$$\frac{8x+7}{x^2-2x-3} = \frac{8x+7}{(x-3)(x+1)}$$

$$= \frac{A}{x-3} + \frac{B}{x+1}$$ for some numbers A,B

There are several methods for finding A, B.

Suppose $\frac{8x+7}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$$\Rightarrow A + \frac{B(x-3)}{x+1} = \frac{8x+7}{x+1}$$

Substitution and algebraic method

Set $x=3$, then I get

$$A + 0 = \frac{8\times3+7}{3+1} = \frac{31}{4}$$

Similarly $$\frac{A(x+1)}{x-3} + B = \frac{8x+7}{x-3}$$

set $x=-1$

$$0 + B = \frac{-8+7}{-1-3} = \frac{1}{4}$$

Thus $$\frac{8x+7}{x^2-2x-3} = \frac{31}{4}\frac{1}{x-3} + \frac{1}{4}\frac{1}{x+1}$$

$$\int \frac{x^3+x+1}{x^2-2x-3}\,dx = \int \left[x+2+\frac{31}{4}\,\frac{1}{x-3}+\frac{1}{4}\,\frac{1}{x+1}\right]dx$$

$$= \frac{x^2}{2}+2x+\frac{31}{4}\ln|x-3|+\frac{1}{4}\ln|x+1|+C$$

Step 2: Starting with $\frac{f}{g}$ with $\deg(f) < \deg(g)$.

$\boxed{\text{We factor } g.}$ How?

Theoretical result: Every real polynomial factors as a product of linear terms and irreducible quadratic terms.

i.e. $$g = \alpha(x-a_1)^{n_1}\cdots(x-a_k)^{n_k}\,(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_\ell x+c_\ell)^{m_\ell}$$

where $b_i^2 < 4c_i$ for all of the quadratic terms this decomposition is unique to reordering

$$x^2 - 2x - 3 = (x-3)(x+1)$$ unique factorisation

eg. $5(x-1)^2(x+1)(x^2+4)^2(x^2+x+1)$

is fully factored and fully grouped

x^2+4 has no real roots

eg. $b^2 - 0^2 < 16 = 4c$

similarly x^2+x+1

has $b^2 = 1 < 4 = 4c$

Step 3:

If we have $$\frac{f}{\alpha(x-a_1)^{n_1}\cdots(x-a_k)^{n_k}(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_\ell x+c_\ell)^{m_\ell}}$$

we can write this a linear combination of

$$\frac{1}{x-a_i}, \frac{1}{(x-a_i)^2}, \ldots, \frac{1}{(x-a_i)^{n_i}}$$

$$\frac{1}{x^2+b_jx+c_j}, \ldots, \frac{1}{(x^2+b_jx+c_j)^{m_j}}$$

$$\frac{x}{x^2+b_j x+c_j}, \dots, \frac{x}{(x^2+b_j x+c_j)^{m_j}}$$

Example:

$$\frac{2x^3}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

There exist A, B, C, D so that this holds.

$$\frac{2x^3}{x^2+1} = \frac{(Ax+B)(x-1)^2}{x^2+1} + C(x-1) + D$$

Set $x=1$

$$\frac{2}{2} = 0 + 0 + D \Rightarrow D = 1$$

$$\frac{2x^3}{(x^2+1)(x-1)^2} - \frac{1}{(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$= \frac{2x^3 - (x^2+1)}{(x^2+1)(x-1)^2} = \frac{2x^3 - x^2 - 1}{(x^2+1)(x-1)^2}$$

$$= \frac{(x-1)(2x^2+x+1)}{(x^2+1)(x-1)^2}$$

$$\Rightarrow \frac{2x^2+x+1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$\Rightarrow \frac{2x^2+x+1}{x^2+1} = \frac{(Ax+B)(x-1)}{x^2+1} + C$$

sot $x=1$

$$\frac{2+1+1}{2} = 0 + C \Rightarrow C = 2$$

$$\Rightarrow \frac{2x^2+x+1}{(x^2+1)(x-1)} - \frac{2}{x-1} = \frac{Ax+B}{x^2+1}$$

$$= \frac{2x^2+x+1-2(x^2+1)}{(x^2+1)(x-1)}$$

$$= \frac{x-1}{(x^2+1)(x-1)} = \frac{1}{x^2+1} = \frac{Ax+B}{x^2+1}$$

$$\Rightarrow A=0 \qquad B=1$$

$$\int \frac{2x^3}{(x^2+1)(x-1)^2}\,dx = \int \left[\frac{1}{x^2+1} + \frac{2}{x-1} + \frac{1}{(x-1)^2}\right] dx$$

$$\int \frac{1}{x^2}\,dx = \frac{-1}{x} + c$$

$$= \arctan(x) + 2\log|x-1| - \frac{1}{x-1} + C$$

$$\frac{3x^2+2}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

Do the algebra (ZZZ)

$$= \frac{5}{x+1} + \frac{-2x-3}{x^2+x+1}$$

$$(x^2+x+1) = 2x+1$$

$$-2x-3 = A(2x+1)+B$$

$$= -(2x+1) - 2$$

$$\int \frac{-2x-3}{x^2+x+1}\,dx = \int \frac{-(2x+1)}{x^2+x+1}\,dx$$

$$-2\int \frac{1}{x^2+x+1}\,dx$$

$$-\int \frac{2x+1}{x^2+x+1}\,dx \quad \left[\begin{array}{l} \text{Let } u = x^2+x+1 \\ du = (2x+1)\,dx \end{array}\right]$$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|x^2+x+1| + C$$

$$-2\int \frac{1}{x^2+x+1}\,dx = -\frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$

$$\int \frac{-2x-3}{x^2+x+1}\,dx = -\ln|x^2+x+1| - \frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$

$$\frac{8x+7}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

Equating coefficients method

$$= \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$$

$$\left[\begin{array}{l} A+B=8 \\ A-3B=7 \end{array}\right. \qquad = \frac{(A+B)x+(A-3B)}{(x-3)(x+1)}$$

$$A = 8 - B$$

$$A+B=8$$

$$(8-B)-3B=7$$

$$A = 8 - \tfrac{1}{4} = \frac{32}{4} - \frac{1}{4}$$

$$= \frac{31}{4}$$

$$8-4B=7$$

$$4B=1 \Rightarrow B = \tfrac{1}{4}$$

$$\left. \cos^2 x + \sin^2 x = 1 \right]$$

$$\Rightarrow \left. 1 + \tan^2 x = \sec^2 x \right]$$

$$\Rightarrow \tan^2 x = \sec^2 x - 1$$