Calculus B, June 1st (Lecture 3)

Proviso from L1: I'm curving to the stratosphere.

Expected average: 50%   Actual: ???

40 is a pretty good grade

I will not destroy your GPA.

Expect tough grading pre-curving.

(☆) Answers uploaded in 24h.   Grading: by the end of the week.

Mirror the style of my solutions.

HW 2 will be up by midnight. (HW 2 due same time nextweek.)

Notes will posted soon (probably partial, to be updated).

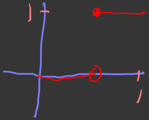
Today: Tiny bit of theory, lots of calculations!!

We defined $\int_a^b f(x)\,dx = K$ when there exists a sequence of partitions P_n of $[a,b]$ such that $\lim_n L(f, P_n) = \lim_n U(f, P_n) = K$.

Which functions are integrable?

Showed $f(x) = x^2$ integrable on $[0,1]$.

On the HW we considered $f(x) = \begin{cases} 0 & x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$



← This is also integrable.

But consider $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$ impossible to graph.

if $[x_i, x_{i+1}]$ has $x_i < x_{i+1}$ then this interval has both rational and irrational #s.

So if P is any partition of $[0,1]$, all of the upper rectangles have height 1, i.e. $U(f,P)=1$.

Similarly $L(f,P)=0$.

There is no hope of bringing these together.

So this f is not integrable!

Sufficient conditions for integrability:

Result: If f is cts. on $[a,b]$, then f

is integrable on $[a,b]$.

Remark: If f is cts. on $[a,b]$, then by the EVT, f is bounded on $[a,b]$.

(sanity check - we only integrate bounded functions – (for now))

The Fundemental Theorem of Calculus

This has two parts. We'll do part I today.

FTC Part I is the most important theorem in Calc B.

But first, let's review the most important thm of Calc A.

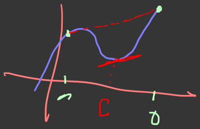
This is of course... Mean Value Theorem.

This is very overpowered.

Thm (MVT): Suppose f is defined on $[a,b]$ and f is cts on $[a,b]$, and differentiable on (a,b) then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{}.$$

$b - a$



One restatement of the conclusion is

$$f(b) = f(a) + f'(c)(b-a)$$

This linear approximation correctly calculates $f(b)$.

Onto FTC Part I.

First a remark. Suppose f is integrable on $[a,b]$ and that K is some value such that for any partition P, we have

$$L(f,P) \le K \le U(f,P).$$

Then we know that $K = \int_a^b f(x)\,dx$.

Why? because if we have P_n s.t.

$$\lim_n L(f,P_n) = \lim_n U(f,P_n)$$

then since each $L(f,P_n) \le K$

we have $\lim_n L(f,P_n) \le K$

and similarly $K \le \lim_n U(f,P_n)$.

This says we can show $\int_a^b f(x)dx = K$ by showing $L(f,P) \le K \le U(f,P)$ for all P.

Main Result (FTC Part I)

Suppose that f is integrable on $[a,b]$ and that F is a function such that F is continuous on $[a,b]$ and such that $F' = f$ on (a,b), then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Twist: This course is about finding F.

This says that if we can find a cts. F such that $F' = f$, then we know $\int_a^b f(x)dx$.

Example: Let $n \in \mathbb{N}$ be any natural number and consider $f(x) = x^n$. Then consider

$F(x) = \frac{x^{n+1}}{n+1}$. Notice that $F'(x) = \frac{(n+1)x^n}{n+1} = x^n$.

Therefore by FTC Part 1, $\int_0^1 x^n dx = F(1) - F(0)$

$$= \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}$$

$$= \frac{1}{n+1}.$$

Doing $\int_0^1 x^2 dx$ using Riemann sums was hard. Now we have a one line solution!!

A continuous F such that $F' = f$ is called an anti-derivative of f.

FTC-I says that if we can find an anti-derivative of F then we already know $\int_a^b f(x)\,dx = F(b) - F(a)$.

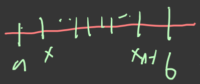
Not every integrable function has an anti-derivative, but FTC Part II will say that this happens most of the time (specifically for cts. functions). - Later

Proof: Suppose that f and F are as hypothesised in the theorem. So f is integrable, F is cts., and $F' = f$ on (a,b).

We will show that for any partition P, we have $L(f,P) \le F(b) - F(a) \le U(f,P)$.

To this end let $P = (\overset{a}{\overset{\parallel}{x_0}} < x_1 < \dots < \overset{b}{\overset{\parallel}{x_n}})$, then

$$F(b) - F(a) = \Big(F(x_n) - F(x_{n-1})\Big) + \Big(F(x_{n-1}) - F(x_{n-2})\Big)$$



$P = (x_0 < x_1 < x_2)$

$F(b) - F(a)$

$= F(x_2) - F(x_0)$

$= (F(x_2) - \cancel{F(x_1)})$

$+ (\cancel{F(x_1)} - F(x_0))$

$+ \dots + (F(x_1) - F(x_0))$

(this is a telescoping sum)

$$= \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

By the MVT there is some $t_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1})$

$= f(t_i)(x_i - x_{i-1})$.

Because $F' = f$.

$$F(b) - F(a) = \sum_{i=0}^{n} (x_i - x_{i-1}) f(t_i)$$

$$U(f, P) = \sum_{i=0}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f$$

$$L(f, P) = \sum_{i=0}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f$$

We have $\inf_{[x_{i-1}, x_i]} f \leq f(t_i) \leq \sup_{[x_{i-1}, x_i]} f$.

So $L(f, P) \leq F(b) - F(a) \leq U(f, P)$.

Therefore $\int_a^b f(x)\,dx = F(b) - F(a)$. ∎

The rest of this course is going to be about finding anti-derivatives.

First off, let's take an inventory of differentiation from Calc A:

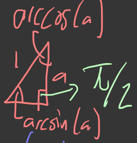
This is everything you learned (⇒ quotient rule) about derivatives in Calc A.

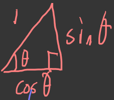
- chain rule (⇒ quotient rule)
- product rule
- derivatives of:
  - i) trig functions (sin, cos, tan, csc, sec, cot)
  - ii) inverse-trig functions (arcsin, arctan)
  - iii) log and e^x
  - iv) x^r power functions
- the facts that $(cf)' = cf'$

  $(f+g)' = f' + g'$

|| derivatives committed to memory

π





$\arccos + \arcsin = \frac{\pi}{2}$

$\arccos = \frac{\pi}{2} - \arcsin$

In essence, we have some basic functions (e.g. power, trig, exp and log) and we know their derivative (by memory).

Then the rest of the functions you consider are built out of basic functions:

i) by linear combinations e.g. $c_1 f + c_2 g$

ii) by multiplying $(f \cdot g)(x) = f(x)g(x)$

iii) by composing $(f \circ g)(x) = f(g(x))$

eg. $f(x) = \sin(x)\left(x^2+5\right)^{10}$

$$f'(x) \overset{\text{product}}{=} \cos(x)\left(x^2+5\right)^{10} + \sin(x)\left(\left(\left(x^2+5\right)^{10}\right)'\right)$$

$$\overset{\text{chain}}{=} \cos(x)\underbrace{\left(x^2+5\right)^{10}}_{} \rightarrow (f \circ g)(x) \quad \text{let } f(x) = x^{10} \quad g(x) = x^2+5$$

$$+ \sin(x)\, 10\left(x^2+5\right)^{9} 2x$$

$$= \cos(x)(x^2+5)^{10}$$

$$+ 20x\sin(x)(x^2+5)^9.$$

This is how differentiation works.

Common mistake: $(f\cdot g)' \neq f'\cdot g'$

(in fact $(f\cdot g)' = f'\cdot g + f\cdot g'$)

In this course we want to undo this process.

Product rule

Chain rule

$$\boxed{\begin{aligned}(f\cdot g)' &= f'\cdot g + f\cdot g'\\ (f\circ g)' &= (f'\circ g)\cdot g'\end{aligned}}$$

Or using x:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(f(g(x)))' = f'(g(x))g'(x)$$

Integration is actually very tricky

- doing the process backwards is not straightforward.

Requires cleverness & insight.

Lots of tricks. Learning this is hard.

Indefinite integral

Suppose that f and that F is an anti-derivative of f (i.e. $F' = f$)

then we write $\int f(x)\,dx = F(x)$

(notice that there are no bounds)

This is the indefinite integral.

Notice that if $F' = f$ and C is a constant then $(F+C)' = f$

so $F+C$ is also an anti-derivative.

Conversely if $F' = G' = f$

then $(F-G)' = 0$

and by the MVT we know $F-G = C$.
(if restricted to an interval)

Suppose $H' = 0$ on $[a,b]$, then

if $H(x) \neq H(y)$ for $x, y \in [a,b]$

then $H'(t) = \frac{H(y)-H(x)}{y-x} \neq 0$. #

(most of theorem.)

Global conspiracy by calculus teachers that the only correct thing to write is $\int f(x)\,dx = F \underline{\underline{+C}}$

Allways write "+C" at the end of indefinate integrals.

Why? It's a meme. (You loose points otherwise.]

Techniques For Integration

1. Integration by inspection.

In general you can find an antiderivative by pulling it out of thin air. Just claiming that $F' = f$.

e.g., $F = \sin(x)\,(x^2+5)^{10}$

notice $F' = \cos(x)(x^2+5)^{10} + 20x\sin(x)(x^2+5)^9$

so $\int \left[\cos(x)(x^2+5)^{10} + 20x\sin(x)(x^2+5)^9\right]dx = \sin(x)(x^2+5)^{10}.$

This violates the "show your work" principle.

This will not get you credit, but it works.

However there are some basic integrals just as there are basic functions:

These are those that correspond to the ll derivatives of basic functions:

$(x^r)' = r x^{r-1} \Rightarrow \boxed{\int x^r dx = \frac{x^{r+1}}{r+1} + C}$

for $r \neq -1$

$$\log'(x) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \log(x) + C$$

Not general
small domain

except the domain of log is $x > 0$

consider $\left[\log(-x)\right]' = \frac{-1}{-x} = \frac{1}{x}$

the domain of $\log|x|$ is $\mathbb{R} \setminus \{0\}$

and $(\log|x|)' = \frac{1}{x}$

more generally $\boxed{\int \frac{1}{x}\,dx = \log|x| + C}$

$$(e^x)' = e^x \qquad \boxed{\int e^x\,dx = e^x + C}$$

trig integrals:

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = \csc(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

6/11 formulas

$$\arctan'(x) = \frac{1}{1+x^2} \qquad \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

2

11 basic integrals always solved by inspection.

2. Linearity of integration

Know $(F+G)' = F' + G'$

$\int f(x)\,dx = F \Leftarrow F' = f \quad G' = g \Rightarrow \int g(x)\,dx = G$

then $\int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

also $\int c f(x)\,dx = c \int f(x)\,dx$

(c must be a constant)

Remember $\int (f \cdot g)(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx$

$\int_0^1 x^2\,dx = \frac{1}{3}$

$\int_0^1 x\,dx = \frac{1}{2}$

$$\boxed{\left(\int_0^1 x\,dx\right)\left(\int_0^1 x\,dx\right)}$$

N.B. There is no general formula for integrating a product.

$$= 1/4 \neq 1/3$$

$$= \int_0^1 x^2 dx$$

Consequence of linearity: integrate any poly

$$\int (x^2 + 5x + 2) dx$$

$$= \int x^2 dx + 5 \int x dx + 2 \int dx$$

$$= \frac{x^3}{3} + 5 \frac{x^2}{2} + 2x + C$$

single constant C at the end.

For the next two sections we'll see that

product rule ⇒ integration by parts

chain rule ⇒ integration by substitution

3. Integration by substitution

$$(f\circ g)'(x) = f'(g(x))\,g'(x)$$

Actually, let $F' = f$ and consider

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

Examples: Recognise something of this form.

$$\int \frac{2x}{x^2+1}\,dx$$

[Let $f(x) = 1/x$

$g(x) = x^2+1$

$g'(x) = 2x$

$$= F(g(x)) + C$$

$$= \log|x^2+1| + C$$

$$f(g(x))\,g'(x) = \frac{2x}{x^2+1}$$

$$F(x) = \int f(x)\,dx = \int \frac{1}{x}\,dx = \log|x| + C$$

Notation:

If we encounter $\int f(g(x))\,g'(x)\,dx$

we write $\left[\begin{array}{l} \text{let } u = g(x) \\ du = g'(x)\,dx \end{array}\right]$

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C$$
$$= F(g(x)) + C$$

suppose $\int f(u)\, du = F(u) + C$

$$\boxed{\begin{aligned} &\int \frac{2x}{x^2+1}\, dx \quad \left[\begin{array}{l} \text{let } u = x^2+1 \\ du = 2x\,dx \end{array}\right] = \int \frac{du}{u} \\ &= \log|u| + C = \log|x^2+1| + C \end{aligned}}$$

$$\int \sin^5(x) \cos(x)\, dx \quad \left[\begin{array}{l} \text{let } u = \sin(x) \\ du = \cos(x)\, dx \end{array}\right]$$

$$= \int u^5\, du = \frac{u^6}{6} + C = \frac{\sin^6(x)}{6} + C$$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx \quad \begin{bmatrix} \text{Let } u = \cos x \\ du = -\sin x\,dx \end{bmatrix}$$

$$= \int \frac{-1}{u}\,du = -\log|u| + C \qquad \frac{d\cos(x)}{dx} = -\sin(x)$$

$$= -\log|\cos(x)| + C \qquad \frac{du}{dx} = -\sin(x)$$

$$\int \frac{dx}{x\log(x)} \quad \begin{bmatrix} \text{Let } u = \log(x) \\ du = \frac{1}{x}\,dx \end{bmatrix}$$

$$= \int \frac{du}{u} = \log|u| + C = \log|\log(x)| + C$$

$$x\,dx = d\left(\frac{x^2}{2}\right)$$

$$\int \frac{x}{1+x^4}\,dx \quad \begin{bmatrix} \text{Let } u = x^2 \\ du = 2x\,dx \end{bmatrix} \text{ AHA!}$$

$$= \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan(u) + C$$

$$= \frac{1}{2}\arctan(x^2) + C.$$

$$\int \sqrt{1+e^x}\,dx \qquad \left[\begin{aligned} \text{Let } u &= 1+e^x \\ du &= e^x\,dx \end{aligned}\right]$$

$$= \int \frac{\sqrt{1+e^x}}{e^x}\,e^x\,dx \qquad e^x = u-1$$

$$= \int \frac{\sqrt{u}}{u-1}\,du \qquad \left[\begin{aligned} \text{Let } w &= \sqrt{u} \\ dw &= \frac{1}{2\sqrt{u}}\,du \end{aligned}\right] \qquad w^2 = u$$

$$= \int \frac{2u}{u-1}\,\frac{1}{2\sqrt{u}}\,du \qquad \frac{2u}{u-1} = \frac{2(u-1)+2}{u-1}$$

$$= \int \left(2+\frac{2}{u-1}\right)\frac{1}{2\sqrt{u}}\,du \qquad = 2+\frac{2}{u-1}$$

$$= \int \left(2+\frac{2}{w^2-1}\right)dw$$

surprising

$$= \int \left(2+\frac{1}{w-1}-\frac{1}{w+1}\right)dw \qquad \boxed{\frac{1}{w-1}-\frac{1}{w+1} = \frac{w+1}{w^2-1}-\frac{w-1}{w^2-1}}$$

$$= 2w + \log|w-1|$$

$$-\log|w+1|+C \qquad \boxed{=\frac{2}{w^2-1}}$$

partial fractions

$$=2\sqrt{u}+\log|\sqrt{u}-1| \qquad \left(\log|w-1|\right)'$$

$$-\log|\sqrt{u}+1|+C \qquad =\frac{1}{w-1}$$

$$=2\sqrt{1+e^x}+\log\left|\sqrt{1+e^x}-1\right|$$

$$-\log\left|\sqrt{1+e^x}+1\right|+C$$

If instead we did $u=\sqrt{1+e^x}$

then we would not have to introduce w.

[No looking up assignment problems on the internet.]