Oral exam advice:

1. It is much better to submit nothing for a problem than to submit nonsense.

2. Study the basics first, not what you understand least. Prioritise "owning" some "core" of the material.

3. Be quick to admit what you do not understand. No BS, it does not work on oral exams.

Exam coverage:

i) HW 1 material and HW 5 material aren't really going to be on the exam.

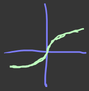
ii) I am a bit overly fond of trig substitutions.

"Basis" questions released on Monday at noon. Due on Wednesday at 11am.

Semi-open book. No external resources whatsoever. Can use the resources on the course website + lecture videos. Can only use methods covered in class.

$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$ $\quad (x-1)^{-2/3} \quad \int x^{-2/3}\,dx = 3x^{1/3}$

$$\lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^{2/3}}\,dx = \lim_{b\to 1^-} 3(x-1)^{1/3}\Big|_0^b$$



$$= \lim_{b\to 1^-}\left[3(b-1)^{1/3} - (-3)\right] = 0 + 3 = 3$$

$$\lim_{b\to 1^+}\int_b^3 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{b\to 1^+} 3(x-1)^{1/3}\Big|_b^3$$

$$= \lim_{b\to 1^-}\left[3\sqrt[3]{2} - 3(b-1)^{1/3}\right] = 3\sqrt[3]{2} - 0$$
$$= 3\sqrt[3]{2}$$

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2}$$

$$\int_0^3 \frac{1}{x-1}\,dx \qquad f(x) = \begin{cases} \log(x) & x>0 \\ \log(-x)+5 & x<0 \end{cases}$$

Naively (wrong!) $= \log|x-1|\Big|_0^3 = \log(2) - \log(1)$
$= \log(2)$

Actually diverges. $(\infty - \infty)$

For $\int_3^5 \frac{1}{(x-1)^{2/3}}\,dx$ Just do the antiderivative.

Comparison tests:

Side remark: If f is integrable on all finite closed subintervals of its domain (e.g. f is continuous, with asymptote)

then $\int_a^\infty f(x)\,dx$ converges

if and only if $\int_a^\infty |f(x)|\,dx$ converges.

Direct comparison test: (Assuming f, g integrable on $[a,b]$)

If $|f(x)| \leq |g(x)|$ on $[a,\infty)$

then a) if $\int_a^\infty g(x)\,dx$ converges

then $\int_a^\infty f(x)\,dx$ converges

b) if $\int_a^\infty f(x)\,dx$ diverges

then $\int_a^\infty g(x)\,dx$ diverges

Does $\int_1^\infty \frac{\sin(x)}{x^2}\,dx$ converge?

Compare this to $\int_1^\infty \frac{1}{x^2}\,dx$

$$\left|\frac{\sin(x)}{x^2}\right| \le \left|\frac{1}{x^2}\right| \text{ because } |\sin(x)| \le 1$$

Now $\int_1^\infty \frac{1}{x^2}\,dx$ converges by the p-test

thus by the direct comparison test, $\int_1^\infty \frac{\sin(x)}{x^2}\,dx$ converges. □

Does $\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}\,dx$ converge?

Compare to $\int_1^\infty \frac{1}{\sqrt{x^2}}\,dx = \int_1^\infty \frac{1}{x}\,dx$

Well, $\sqrt{x^2 - 0.1} \leq \sqrt{x^2}$

so $\left|\frac{1}{x}\right| = \left|\frac{1}{\sqrt{x^2}}\right| \leq \left|\frac{1}{\sqrt{x^2-0.1}}\right|$

Now $\int_1^\infty \frac{1}{x}\,dx$ diverges by the p-test

so by the direct comparison test

$\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}\,dx$ diverges. □

If we considered $\int_1^\infty \frac{1}{\sqrt{x^2+0.1}}\,dx$

then then the direct comparison test does not help.

If f and g are nice, eg cts.

Limit comparison test: v If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$

for $0 < L < \infty$ then

$\int_a^\infty f(x)\,dx$ converges if and only if

$\int_a^\infty g(x)\,dx$ converges.

Eg. $\int_1^\infty \frac{1}{\sqrt{x^2+0.1}}\,dx$

Compare to $\int_1^\infty \frac{1}{\sqrt{x^2}}\,dx$

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2}}{1/\sqrt{x^2+0.1}} = \lim_{x\to\infty} \frac{\sqrt{x^2+0.1}}{\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \sqrt{\frac{x^2+0.1}{x^2}} = \lim_{x\to\infty} \sqrt{1+\frac{0.1}{x^2}}$$

$$= \sqrt{\lim_{x\to\infty}\left(1+\frac{0.1}{x^2}\right)} = \sqrt{1+0} = 1$$

Thus since $\int_1^\infty \frac{1}{x}\,dx$ diverges

so does $\int_1^\infty \frac{1}{\sqrt{x^2+8.1}}$ by the

limit comparison test.

Does $\int_1^\infty \frac{1}{x^2+1}\,dx$ converge?

Compare to $\int_1^\infty \frac{1}{x^2}\,dx$

Consider $\lim_{x\to\infty} \frac{1/x^2}{1/(x^2+1)} = \lim_{x\to\infty} \frac{x^2+1}{x^2}$

$= \lim_{x\to\infty} 1+\frac{1}{x^2} = 1$

So since $\int_1^\infty \frac{1}{x^2}\,dx$ converges by the p-test

so does $\int_1^\infty \frac{1}{x^2+1}\,dx$ by the limit comparison test. $\square$

Review Session:

FTC-I says that if f is integrable $[a,b]$, F is cts. on $[a,b]$ with $F' = f$ on (a,b), then
$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

e.g. finding $\int_0^1 x^2\,dx$ with Riemann sums was hard

However, we can notice $\left(\frac{1}{3}x^3\right)' = x^2$, so by FTC-I $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$.

FTC-I tells us why antiderivatives are important.

$\int_a^b f(x)\,dx$ – definite integral

$\int f(x)\,dx$ – indefinite integral: refers to some antiderivative of f.

How do we find antiderivatives?

1.) By inspection. We can notice that $F' = f$.

You have to memorise ll integrals that come from the derivatives of the basic Calc A functions.

The functions (answers to the integrals of their derivatives) are:

x^r, $\log(x)$, e^x, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$, $\csc(x)$
$\cot(x)$, $\arcsin(x)$, $\arctan(x)$

eg. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$ $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

Later on, we added a few more integrals to the "inspection" list (skip boring substitutions):

$$\int \frac{1}{x+a}\,dx = \log|x+a| + C$$

$r \neq -1$ $$\int (x+a)^r\,dx = \frac{1}{r+1}(x+a)^{r+1} + C$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$$

similarly $$\int \sin(ax)\,dx = \frac{-1}{a}\cos(ax) + C$$

2) Linearity of integration:

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int a f(x)\,dx = a\int f(x)\,dx$$ (a is a constant)

We often move numbers outside of integrals.

3) Substitution: Comes from chain rule.

$$(f \circ g)(x) = f(g(x))$$

$$(f \circ g)'(x) = f'(g(x))\, g'(x)$$

$$f(g(x)) = \int f'(g(x))\, g'(x)\, dx$$

Suppose $F(u) = \int f(u)\, du$

Then $\int f(g(x))\, g'(x)\, dx = F(g(x))\, dx$

In order to find $\int f(g(x))\, g'(x)\, dx$ we first solve $\int f(u)\, du$ and then plug in $u = g(x)$ to find the answer.

We use this with a special relation:

$$\int f(\underbrace{g(x)}_{u})\, \underbrace{g'(x)\, dx}_{du} \begin{bmatrix} \text{Let } u = g(x) \\ du = g'(x)\, dx \end{bmatrix} = \int f(u)\, du$$

For definite integrals, we do not substitute back, instead we change the bounds when we substitute:

$$\int_a^b f(g(x))\,g'(x)\,dx \begin{bmatrix} \text{Let } u = g(x) \\ du = g'(x)\,dx \end{bmatrix} = \int_{g(a)}^{g(b)} f(u)\,du$$

Examples:

$$\int \frac{2x}{x^2+1}\,dx \begin{bmatrix} \text{Let } u = x^2+1 \\ du = 2x\,dx \end{bmatrix} = \int \frac{1}{u}\,du = \log|u| + C$$

$$= \log|x^2+1| + C \quad \square$$

$$= \log(x^2+1) + C$$

$$\int \sin^5(x)\cos(x)\,dx \begin{bmatrix} \text{Let } u = \sin(x) \\ du = \cos(x)\,dx \end{bmatrix}$$

$$= \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6(x) + C \quad \square$$

4. **Integration by Parts** comes from the product rule

$$(f \cdot g)(x) = f(x)g(x)$$

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$f(x)g(x) = \int \left[f'(x)g(x) + f(x)g'(x)\right] dx$$

$$\Rightarrow \int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$u = f(x) \qquad du = f'(x)\,dx$$

$$v = g(x) \qquad dv = g'(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

To apply parts:

$$\int f(x)\,g'(x)\,dx \left[\begin{array}{ll} u = f(x) & dv = g'(x)\,dx \\ du = f'(x)\,dx & v = g(x) \end{array}\right]$$

$$= f(x)\,g(x) - \int g(x)\,f'(x)\,dx$$

Examples:

$$\int x e^{x}\,dx \left[\begin{array}{ll} u = x & dv = e^{x}\,dx \\ du = dx & v = e^{x} \end{array}\right] = xe^{x} - \int e^{x}\,dx$$

$$= xe^{x} - e^{x} + C \qquad \square$$

$$\int \frac{\log|x|}{x}\,dx \left[\begin{array}{ll} u = \log|x| & dv = 1/x\,dx \\ du = 1/x\,dx & v = \log|x| \end{array}\right]$$

$$= \log^2|x| - \int \frac{\log|x|}{x}\,dx$$

$$\Rightarrow 2\int \frac{\log|x|}{x}\,dx = \log^2|x|$$

$$\Rightarrow \int \frac{\log|x|}{x}\,dx = \frac{1}{2}\log^2|x| + C \quad \square$$

5) Partial fractions

Lets you integrate *any* rational function.

A rational function is a ratio of polynomials.

eg. $\displaystyle\int \frac{x^3+x+1}{x^2-2x-3}\,dx$

In general if you have $\frac{f}{g}$ with $\deg(f) \geq \deg(g)$

then by doing a polynomial long division

we can write it as $q + \frac{r}{g}$ where $\deg(r) < \deg(g)$.

$$\begin{array}{r|l} & x+2 \\ \hline x^2-2x-3 & x^3+0x^2+x+1 \\ & \underline{x^3-2x^2-3x} \end{array}$$

$$\begin{array}{r} 2x^2 + 4x + 1 \\ 2x^2 - 4x - 6 \\ \hline 8x + 7 \end{array}$$

$$\frac{x^3 + x + 1}{x^2 - 2x - 3} = x + 2 + \frac{8x + 7}{x^2 - 2x - 3}$$

$$\frac{8x + 7}{x^2 - 2x - 3} = \frac{8x + 7}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-3)}{(x-3)(x+1)} = \frac{(A+B)x + (A-3B)}{(x-3)(x+1)}$$

$$\Rightarrow A + B = 8, \quad A - 3B = 7$$

$$\Rightarrow 8 - B - 3B = 7 \Rightarrow -4B = -1 \Rightarrow B = \frac{1}{4}$$

$$\Rightarrow A = 8 - \frac{1}{4} = \frac{32}{4} - \frac{1}{4} = \frac{31}{4}$$

$$\int \frac{x^3+x+1}{x^2-2x-3}\,dx = \int\left(x+2+\frac{31}{4}\,\frac{1}{x-3}+\frac{1}{4}\,\frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}x^2+2x+\frac{31}{4}\log|x-3|+\frac{1}{4}\log|x+1|+C \quad \square$$

x^2+1 has no roots

in general x^2+bx+c then if $b^2<4c$

then this has no real roots

and is irreducible

A third of you forgot about this on the midterm.

$$\frac{p}{(x+1)^3(x^2+1)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{\boxed{Dx}+E}{x^2+1}$$

You MUST show all of this work!

Completing the square:

I know how to integrate

$\int \frac{1}{1+u^2}\,du$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(x+\frac{1}{2})^2-\frac{1}{4}+1}\,dx$$

x^2+2b

$= (x+b)^2-b^2$

$b^2-4c<0 \Rightarrow$ no roots

$$= \int \frac{1}{(x+\frac{1}{2})^2+3/4}\,dx = \int \frac{1}{3/4\left(\frac{4}{3}(x+\frac{1}{2})^2+1\right)}\,dx$$

$$= \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}(x+1/2)\right)^2+1}\,dx \quad \left[\begin{array}{l} \text{Let } u = \frac{2}{\sqrt{3}}(x+\frac{1}{2}) \\ du = \frac{2}{\sqrt{3}}\,dx \end{array}\right]$$

$$= \frac{4}{3}\,\frac{\sqrt{3}}{2}\int \frac{1}{u^2+1}\,du = \frac{2}{\sqrt{3}}\arctan(u)+C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C \qquad \square$$

Completing the square absorbs linear terms into quadratic terms.

$$\int \frac{1}{(4x^2-24x+27)^{3/2}}\,dx \quad \left[\begin{array}{l} \text{Let } \frac{2}{3}(x-3) = \sec(u) \\ \frac{2}{3}\,dx = \sec(u)\tan(u)\,du \end{array}\right]$$

$$4x^2-24x+27 = 4(x^2-6x)+27$$

$$= 4\left((x-3)^2 - 9\right) + 27 = 4(x-3)^2 - 9 = (*)$$

$$= 9\left(\frac{4}{9}(x-3)^2 - 1\right) = 9\left(\left(\frac{2}{3}(x-3)\right)^2 - 1\right)$$

$$(*) = 4\left((x-3)^2 - \frac{9}{4}\right)$$

let $x-3 = \frac{3}{2}\sec(u)$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

eg. $\sec^2(x) - 1 = \tan^2(x)$

$\cos^2(x) = 1 - \sin^2(x)$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$= 1 - 2\sin^2(x)$$
$$= 2\cos^2(x) - 1$$

$\Rightarrow$ $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$\sin^2(x) = \frac{1}{2}(1-\sin(2x))$

$\int \cos^2(x)\,dx$ use $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$$\int \cos^3(x)\,dx = \int \left(1-\sin^2(x)\right)\cos(x)\,dx \quad \left[\begin{array}{l} \text{let } u = \sin(x) \\ du = \cos(x)\,dx \end{array}\right]$$

$$= \int (1-u^2)\,du$$ ✓

300 IQ sub $u = \tan\left(\frac{x}{2}\right)$

$$\cos(x) = \frac{1-u^2}{1+u^2}, \quad \sin(x) = \frac{2u}{1+u^2}$$

$$dx = \frac{2}{1+u^2}\,du$$

Trig substitutions

If you have $\sqrt{1-x^2}$ I could remove the square root if $x = \sin(u)$

then $\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \sqrt{\cos^2(u)} = \cos(u)$

If I have	use
$a^2 + x^2$	$x = a\tan(u)$
$x^2 - a^2$	$x = a\sec(u)$
$a^2 - x^2$	$x = a\sin(u)$

Pythagoras will apply

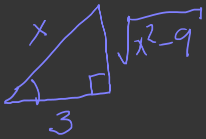
$$\int \frac{dx}{(x^2-9)^{3/2}} \quad \left[\begin{array}{l} \text{Let } x = 3\sec(u) \\ dx = 3\sec(u)\tan(u)\,du \end{array}\right]$$

$$= \int \frac{3\sec(u)\tan(u)}{(9\sec^2(u)-9)^{3/2}}\,du = \frac{3}{9^{3/2}}\int \frac{\sec(u)\tan(u)}{(\sec^2(u)-1)^{3/2}}\,du$$

$$= \frac{1}{9}\int \frac{\sec(u)\tan(u)}{[\tan^2(u)]^{3/2}}\,du = \frac{1}{9}\int \frac{\sec(u)}{\tan^2(u)}\,du$$

$$= \frac{1}{9}\int \frac{1}{\cos(u)}\,\frac{\cos^2(u)}{\sin^2(u)}\,du = \frac{1}{9}\int \csc(u)\cot(u)\,du$$

$$= -\frac{1}{9}\csc(u) + C = -\frac{1}{9}\,\mathrm{ccs}\left(\operatorname{arcsec}\left(\frac{x}{3}\right)\right) + C$$



$$= -\frac{1}{9}\,\frac{x}{\sqrt{x^2-9}} + C \quad \square$$