MAT 126: CALCULUS B

Greek geometry, the notion of area.

Definitions:

1. A point is that which has no part.
2. A line is a breadthless length.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all straight lines falling upon it from one point equal one another.
16. And the point is called the centre of the circle.

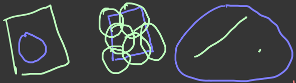
Common Notions:

- Things which are equal to the same thing also equal one another.
- If equals are added to equals, the wholes are equal.
- If equals are subtracted from equals, the remainders are equal.
- Things which coincide with one another equal one another.
- The whole is greater than the part.

Postulates:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and radius.
4. That all right angles equal one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which are the angles less than two right angles.

Theory of proportions: (Eudoxus)



V Def. 4: Two magnitudes are said to have a ratio to one another when multiples of each can excede the other.

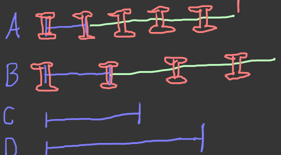
Def. 5: Magnitudes are said to be in the same ratio, "the first to the second as the third to the fourth," when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike excede, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

"A is to B as C is to D"

$A:B :: C:D$

Modern terms

$\frac{A}{B} = \frac{C}{D}$



$A < B$ $2B > 3A$

$2A > B$

$C < D$

$2C > D$

compare nA to mB

$nA \; ? \; mB$

? is $<$ $>$ $=$

X Prop 1.

(The Exhaustion Lemma)

Two unequal magnitudes being set out,* if from the greater is subtracted a magnitude greater than its half, and from this a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out.

* magnitudes that have a ratio

Real numbers > 0

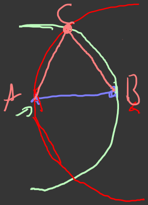
1024 511 253 120 55 20 7 $\boxed{3}$

∧

3,14…

$\pi \simeq 3.14..$

I Prop 1: To construct an equilateral triangle on a given finite straight line.



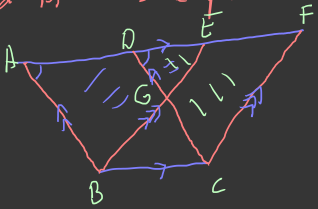
$\overline{AB} = \overline{CB}$

$\overline{AB} = \overline{AC}$

$\overline{CB} = \overline{AC}$

∎

Prop 35: Parallelograms which share the same base and in the same parallels equal one another.



□ ABCD = □ BEFC

$\overline{AD} = \overline{BC} = \overline{EF}$

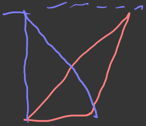
$\overline{AE} = \overline{DF}$

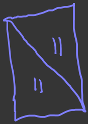
$\overline{BA} = \overline{DC}$

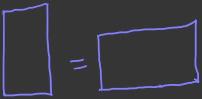
$\Delta ABE = \Delta CDF$

subtract DGE

add BGC ▨

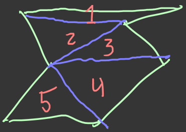


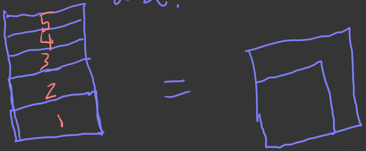




Given a rectangle, to construct an equal rectangle on any other base.

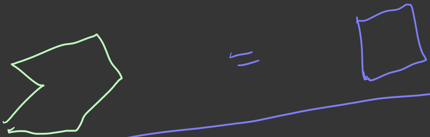
Rectilinear areas





← fix this base

Moral: Given any rectilinear figure to construct a square of equal area.



c. 500 BC

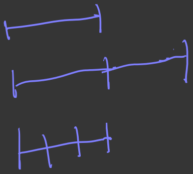
Foundational crises

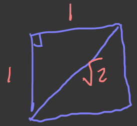
1. Compatibility of curvilinear with rectilinear

areas.

2. Pythagorean theory of proportions.

sort of a theory of $\frac{a}{b}$ rational numbers.





$\sqrt{2}$ is not $\frac{a}{b}$ for $a, b \in \mathbb{N}$

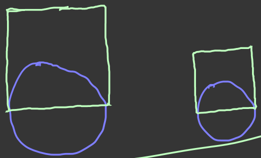
The proportion of the diagonal to the side of a square was not admissible.

× 3. Squaring the circle? Thought this was impossible.

Central problem:

Conjecture: Two circles are to one another as the squares on their diameters.

✓

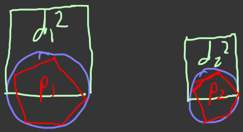


Hero of the story: Eudoxus

Solves everything in two pages!

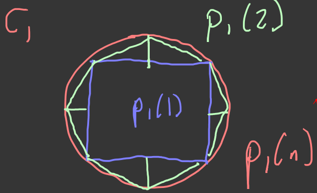
Eudoxus's Proof

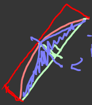
Lemma: (XII Prop. 1) Similar polygons inscribed in circles are to one another as the squares on the diameters of the circles.



$$\frac{p_1}{p_2} = \frac{d_1^2}{d_2^2}$$ □

Theorem: Circles are as the squares on their diameters.

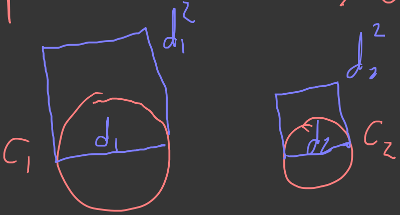




is half of the rectangle

$C_1 - p_1(1)$ in other words $\frac{1}{2}[C_1 - p_1(n)]$

$C_1 - p_1(2)$ $> C_1 - p_1(n+1)$



Claim: (Goal)

$$\frac{C_1}{C_2} = \frac{d_1^2}{d_2^2}$$

$p_1(n)$ similar to $p_2(n)$ Know $\frac{d_1^2}{d_2^2} = \frac{p_1(n)}{p_2(n)}$

Suppose C_1 is to some area s

as d_1^2 is to d_2^2

$$\frac{C_1}{s} = \frac{d_1^2}{d_2^2}$$

Consider s vs. C_2

Cases: $s < C_2$ ✗ will show that assuming $s < C_2$, I get a contrad.

$s > C_2$ ✗

$s = C_2$ ✓

Suppose $\boxed{s < C_2}$

$C_2 - s$ is a non-zero area

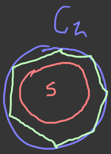
$[C_2 - p_2(n)$ By the exhaustion lemma

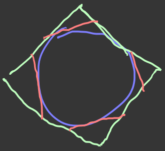
I get some n so that

$$C_2 - p_2(n) < C_2 - s$$

$$s < p_2(n) < C_2$$

$\Rightarrow$





$$\frac{C_1}{S} = \frac{d_1^2}{d_2^2} = \frac{p_1(n)}{p_2(n)} \qquad \frac{C_1}{p_1(n)} = \frac{}{p_2(n)}$$

$p_1(n) < C_1$

Pirate books here

libgen

$\Rightarrow \boxed{p_2(n) < S}$ by the def'n of two ratios being equal.

"$A = \pi r^2$"

A is a constant r^2

But I had $\boxed{S < p_2(n)}$ #

▨

Resume at 7:45

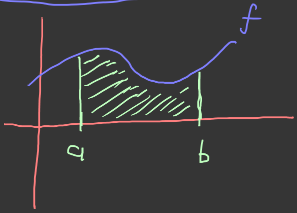
Onto Calculus: (Course starts now.)

[**Promise:** This course is going to be curved into the stratosphere. □

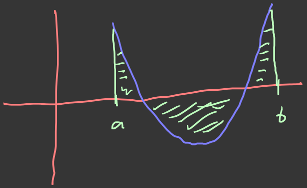
[Chegg is banned. Like super hard banned]

[**Personal objective:** Get you all to write high quality solutions to mathematical problems.

By appointment office hours.



Central problem:
Find the area bound by the x-axis and some function.
This area is called "the area under the curve".



Convention: area under the x axis is negative.
In other words we want the difference of the area above with the area below.

Sets are collections of things.
In math every type of thing lives in some set.

e.g. $\mathbb{N}$ the set of natural numbers

$\{0, 1, 2, 3, \dots\}$

is an element of

$0 \in \mathbb{N}$

$\mathbb{Z}$ the set of integers

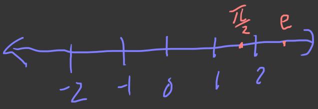
$\{\dots, -2, -1, 0, 1, 2, \dots\}$ $-2 \in \mathbb{Z}$

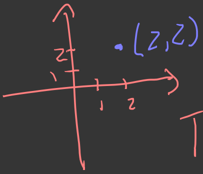
$\mathbb{Q}$ the set of rational numbers

$\frac{1}{2}, \frac{3}{2}, -2 \in \mathbb{Q}$ $\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \}$

$\mathbb{R}$ the set of real numbers

We think of these as points on a number line:





There is a notion of ordered pairs

there is thus set $\mathbb{R} \times \mathbb{R}$ ↳ means the set of pairs

$(2,2) \in \mathbb{R} \times \mathbb{R}$

In general if A and B are sets, I have

$(\frac{1}{2}, 2) \in \mathbb{Q} \times \mathbb{N}$ $A \times B$ the set of pairs with first component in A and second component in B.

$A \times B = \{ (x,y) : x \in A, y \in B \}$

If $x \in A \Rightarrow x \in B$ for all x.

In other words if every element of A is an element of B then we write

$A \subseteq B$ and say A is a subset of B.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

We have a notion of subsets and so there should be a set of all subsets.

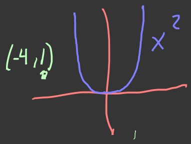
Let $\mathcal{P}(A)$ denote the set of all subsets of A,

$$\{B : B \subseteq A\}$$

What is the relationship between:

$$\boxed{y = x^2} \qquad \boxed{f(x) = x^2}$$

↑ relations   functions

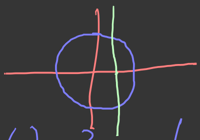


A relation is a set of points in the plane.

$$\{(x,y) : y = x^2\} \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$$

$$\subseteq \mathbb{R} \times \mathbb{R}$$

$$\boxed{x^2 + y^2 = 1}$$



vertical line test

$y = \sqrt{1-x^2}$ ← not the whole circle.

I can't write $f(x) = ?$ and get the circle.

$$\{(x,y) : x^2 + y^2 = 1\}.$$

Ordering:

On $\mathbb{R}$ we have a notion of $\leq$

e.g. $1 \leq 2$, $2 \leq 4$, $4 \not\leq 1$.

We can think of this as a relation. (Defined by the set of all (x, y) with $x \leq y$.)

What are some properties of $\leq$?

1. Reflexivity: $x \leq x$
2. Transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$.
3. Anti-symmetry: if $x \leq y$ and $y \leq x$, then $x = y$
4. Totality: for any x, y, either $x \leq y$ or $y \leq x$

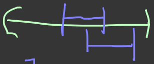
These four properties define a total order.

e.g. $\leq$ on $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$

If we drop the fourth condition we get the notion of a partial order.

e.g. $\mathcal{P}(A)$ has an order given by $\subseteq$.

$\mathcal{P}(\mathbb{R})$ $[0,1] \subseteq [-1,4]$



because $[0,2]$ and $[1,3]$ cannot be compared.

For this course we will order $\mathcal{P}(A)$ by reverse containment. Meaning that $B \le C$ means $C \subseteq B$,

while $[0,2]$ and $[1,3]$ cannot be compared, we have $\emptyset \subset [0,2]$ $\emptyset \subseteq [1,3]$

↳ empty set

$[0,2] \le \emptyset$ $[1,3] \le \emptyset$

5. Directedness: For all x, y there is some z so that $x \le z$ and $y \le z$.

If we have a directed partial order we call it a directed set.

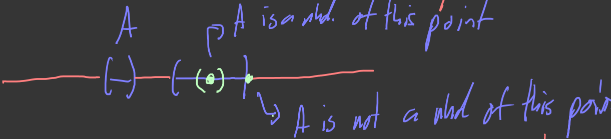
$(x,y) = \{z : x < z < y\}$

Important definition:

If I fix $a \in \mathbb{R}$ some real number, then we say that a set $A \subseteq \mathbb{R}$ is a neighbourhood of a if

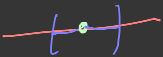
informal: if $a \in A$ and a has some "wiggle room"

formal: if there is some interval $(x,y) \subseteq \mathbb{R}$ so that $a \in (x,y) \subseteq A$.



A is a nbd. of this point

A is not a nbd of this point.



I will denote the set of all neighbourhoods of a by $\mathscr{O}(a)$

script O

$\mathscr{O}(a)$ is ordered by reverse inclusion.

Suppose $\underline{A}, \underline{B} \in \underline{\mathscr{O}}(a)$

intervals

This means $a \in (x_1, y_1) \subseteq A$

means $x_1 < a < y_1$

$a \in (x_2, y_2) \subseteq B$

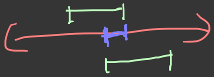
$x_2 < a < y_2$

Then $\max(x_1, x_2) < a < \min(y_1, y_2)$

so $\boxed{a \in (\max(x_1,x_2), \min(y_1,y_2)) \subseteq A \cap B}$

$A \cap B = \{x : x \in A, x \in B\}$ intersection

$[0,2] \cap [1,3] = [1,2]$



This shows that $A \cap B$ is a neighborhood of a.

In other words $\boxed{A \cap B \in \mathcal{O}(a).}$

Notice $A \cap B \subseteq A$

$A \cap B \subseteq B$

means $A \leq A \cap B$

$B \leq A \cap B$

So $\mathcal{O}(a)$ is a directed set.

Much more interesting than $\mathcal{P}(A)$.

Limits:

In Calc A you saw $\lim\limits_{x \to a} f(x)$

In Calc C we will use $\lim\limits_{n \to \infty} x_n$

the limit of a sequence.

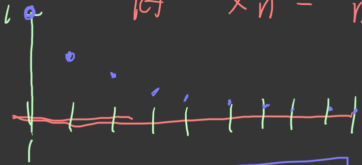
we'll discuss this first

A sequence is for every natural number, some real number.

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$$

By this I mean that for every $n \in \mathbb{N}$,

let $x_n = \frac{1}{n+1}$.



As n gets big $\frac{1}{n}$ gets small.

We should have

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Why?

Because as n gets big $\frac{1}{n}$ is eventually abitrarilly small.

The word "eventually" reffers to a directed set.

And the word "abitrarilly" is parametrised by the neighbourhoods of a point.

$\mathbb{N}$ has an order $\leq$.

Eventually means that some condition holds "eventually" in the sense that there is some N st the condition is met when $n \geq N$.

Arbitrarilly close to 0 means that x_n is eventually in any neighbourhood of 0. Close admits wiggle room. So neighbourhoods are appropriate.

Def: We say $x_n \to \ell$, for some $\ell \in \mathbb{R}$, if for all $A \in \mathcal{U}(\ell)$ we have some $N \in \mathbb{N}$ so that $x_n \in A$ whenever $n \geq N$.