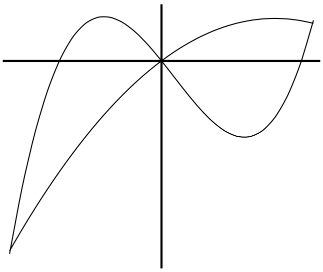
MAT 126: Problem Set 4 Solutions

1. We have $f(x) = -x^2 + 3x$, and $g(x) = 2x^3 - x^2 - 5x$. We first solve for the intersection points:

$$\begin{aligned} & -x^2 + 3x = 2x^3 - x^2 - 5x \\ \Rightarrow \quad & 2x(x-2)(x+2) = 0 \\ \Rightarrow \quad & x = 0, \pm 2 \end{aligned}$$

Looking at the equations, we see that one is a parabola that opens downwards, while the other is a cubic with a leading term of $2x^3$. We may also easily factor each equation to obtain its roots. This information is sufficient to produce a rough sketch of the curves:

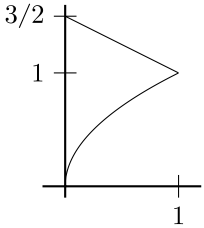


We see that the $g \geqslant f$ on $[-2, 0]$ while $f \geqslant g$ on $[0, 2]$. From this, we can set up an integral which computes the area:

$$\begin{aligned} A &= \int_{-2}^{0} [g(x) - f(x)]\, dx + \int_{0}^{2} [f(x) - g(x)]\, dx \\ &= \int_{-2}^{0} \left[2x^3 - 8x\right] dx + \int_{0}^{2} \left[8x - 2x^3\right] dx \\ &= \left(\frac{1}{2}x^4 - 4x^2\right)\Bigg|_{-2}^{0} + \left(4x^2 - \frac{1}{2}x^4\right)\Bigg|_{0}^{2} \\ &= [0 - (8 - 16)] + [(16 - 8) - 0] = 16 \end{aligned}$$

□

2. By inspection, when $x = 1$, we have $3/2 - 1/2 = \sqrt{1}$. These curves are easy to sketch:



a) To set up an integral with respect to x that computes the area, we simply integrate the lengths of the vertical slices of this region over the interval $[0, 1]$.

$$\begin{aligned}
A &= \int_0^1 \left[\left(\frac{3}{2} - \frac{1}{2}x \right) - \sqrt{x} \right] dx \\
&= \left(\frac{3}{2}x - \frac{1}{4}x^2 - \frac{2}{3}x^{3/2} \right) \Big|_0^1 \\
&= \left(\frac{3}{2} - \frac{1}{4} - \frac{2}{3} \right) - 0 \\
&= \frac{18 - 3 - 8}{12} = \frac{7}{12}
\end{aligned}$$

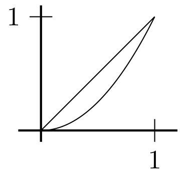
□

b) To set up an integral with respect to y that computes the area, we have to split the region up into pieces. Solving for x, our equations become $x = 3 - 2y$ and $x = y^2$. Writing an integral for the area, we obtain:

$$\begin{aligned}
A &= \int_0^1 y^2 \, dy + \int_1^{3/2} [3 - 2y] \, dy \\
&= \frac{1}{3}y^3 \Big|_0^1 + \left(3y - y^2\right) \Big|_0^{3/2} \\
&= \left(\frac{1}{3} - 0 \right) + \left(\left(\frac{9}{2} - \frac{9}{4} \right) - (3 - 1) \right) \\
&= \frac{4 + 27 - 24}{12} = \frac{7}{12}
\end{aligned}$$

□

3. As always, we begin with a sketch:



a) We will use the washers method to find the volume of the solid of revolution.

$$\begin{aligned}
V &= \int_0^1 \pi \left[(x)^2 - \left(x^2\right)^2 \right] dx \\
&= \pi \int_0^1 \left[x^2 - x^4\right] dx = \pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^1 \\
&= \pi \left(\frac{1}{3} - \frac{1}{5} \right) - 0 = \frac{2}{15}\pi
\end{aligned}$$

□

b) For this part, we use cylindrical shells. We have $h(x) = x - x^2$.

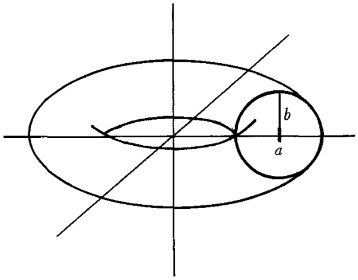
$$\begin{aligned}
V &= \int_0^1 2\pi x h(x)\, dx = \int_0^1 2\pi x \left(x - x^2\right)\, dx \\
&= 2\pi \int_0^1 \left[x^2 - x^3\right] dx = 2\pi \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 \\
&= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) - 0 = \frac{1}{6}\pi
\end{aligned} \qquad \square$$

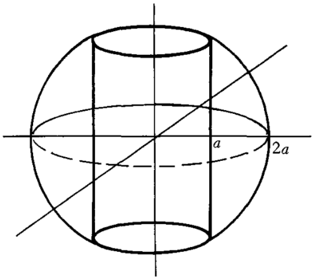
4. We will find the volume of the torus using the cylindrical shells method while integrating with respect to x. First picture the region inside of the curve $(x-a)^2 + y^2 = b^2$. The range of x values over which there lies a vertical slice of the circle is $[a-b, a+b]$. We would like to find the heights of these slices. Towards this end, we solve for y when x is taken as fixed:

$$\begin{aligned}
&(x-a)^2 + y^2 = b^2 \\
&\Rightarrow y^2 = b^2 - (x-a)^2 \\
&\Rightarrow y = \pm\sqrt{b^2 - (x-a)^2} \\
&\Rightarrow h(x) = 2\sqrt{b^2 - (x-a)^2}
\end{aligned}$$

Knowing the heights and the bounds allows us to set up an integral:

$$\begin{aligned}
V &= \int_{a-b}^{a+b} 2\pi x h(x)\, dx = 4\pi \int_{a-b}^{a+b} x\sqrt{b^2 - (x-a)^2}\, dx \quad \left[\begin{array}{c}\text{Let } x - a = b\sin(u) \\ dx = b\cos(u)\, du\end{array}\right] \\
&= 4\pi \int_{\arcsin(-1)}^{\arcsin(1)} (b\sin(u) + a)\sqrt{b^2 - b^2\sin^2(u)}\, b\cos(u)\, du \\
&= 4\pi b^2 \int_{-\pi/2}^{\pi/2} (b\sin(u) + a)\cos^2(u)\, du \\
&= 4\pi b^3 \int_{-\pi/2}^{\pi/2} \cos^2(u)\sin(u)\, du \left[\begin{array}{c}\text{Let } w = \cos(u) \\ dw = -\sin(u)\, du\end{array}\right] + 2\pi a b^2 \int_{-\pi/2}^{\pi/2} [1 + \cos(2u)]\, du \\
&= -4\pi b^3 \int_0^0 w^2\, dw + \left(2\pi a b^2 u + \pi a b^2 \sin(2u)\right)\Big|_{-\pi/2}^{\pi/2} \\
&= 0 + \left(\pi^2 a b^2 + 0\right) - \left(-\pi^2 a b^2 + 0\right) = 2\pi^2 a b^2
\end{aligned} \qquad \square$$





5. We will again use the cylindrical shells method while integrating with respect to x. If we look at the cross section of this solid in the xy-plane, then we get a circle of radius $2a$ with all vertical slices lying over $[-a, a]$ removed. If we parametrise the remaining cylindrical shells (which extend on both sides of the y axis) by x values, then our parameter is restricted to lie in $[a, 2a]$. We solve for the height as before:

$$\begin{aligned} & x^2 + y^2 = (2a)^2 \\ & \Rightarrow y^2 = 4a^2 - x^2 \\ & \Rightarrow y = \pm\sqrt{4a^2 - x^2} \\ & \Rightarrow h(x) = 2\sqrt{4a^2 - x^2} \end{aligned}$$

We set up the integral as before:

$$\begin{aligned} V &= \int_a^{2a} 2\pi x h(x)\, dx = 4\pi \int_a^{2a} x\sqrt{4a^2 - x^2}\, dx \quad \left[\begin{array}{l} \text{Let } x = 2a\sin(u) \\ dx = 2a\cos(u)\, du \end{array}\right] \\ &= 16\pi a^2 \int_{\arcsin(1/2)}^{\arcsin(1)} \sin(u)\sqrt{4a^2 - 4a^2\sin^2(u)}\, \cos(u)\, du \\ &= 32\pi a^3 \int_{\pi/6}^{\pi/2} \cos^2(u)\sin(u)\, du \quad \left[\begin{array}{l} \text{Let } w = \cos(u) \\ dw = -\sin(u)\, du \end{array}\right] \\ &= -32\pi a^3 \int_{\sqrt{3}/2}^{0} w^2\, dw = -\frac{32}{3}\pi a^3 w^3 \Big|_{\sqrt{3}/2}^{0} = 4\sqrt{3}\pi a^3 \end{aligned}$$

□

6. a)

$$f(x) = \frac{1}{24}x^3 + \frac{2}{x}$$
$$f'(x) = \frac{1}{8}x^2 - \frac{2}{x^2}$$
$$1 + (f'(x))^2 = 1 + \left(\frac{1}{8}x^2 - \frac{2}{x^2}\right)^2$$
$$= 1 + \frac{1}{64}x^4 - \frac{1}{2} + \frac{4}{x^4}$$
$$= \frac{1}{64}x^4 + \frac{1}{2} + \frac{4}{x^4} = \left(\frac{1}{8}x^2 + \frac{2}{x^2}\right)^2$$
$$l = \int_2^4 \sqrt{1 + (f'(x))^2}\, dx = \int_2^4 \left(\frac{1}{8}x^2 + \frac{2}{x^2}\right) dx$$
$$= \left(\frac{1}{24}x^3 - \frac{2}{x}\right)\bigg|_2^4 = \left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)$$
$$= \frac{16 - 3 - 2 + 6}{6} = \frac{17}{6}$$

□

6. b)

$$f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$
$$f'(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$1 + (f'(x))^2 = 1 + \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}$$
$$= \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} = \frac{1}{4}\left(e^x + e^{-x}\right)^2$$
$$l = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \frac{1}{2}\left(e^x + e^{-x}\right) dx$$
$$= \frac{1}{2}\left(e^x - e^{-x}\right)\bigg|_0^1 = \frac{1}{2}\left(e - e^{-1}\right)$$

□

7. a)

$$\int_0^{\pi/3} \frac{1}{2 - \cos(x)}\, dx \; \left[\text{Let } u = \tan(x/2)\right] = \int_0^{1/\sqrt{3}} \frac{1}{2 - \frac{1-u^2}{1+u^2}} \frac{2}{1+u^2}\, du$$
$$= \int_0^{1/\sqrt{3}} \frac{2}{2(1+u^2) - (1-u^2)}\, du = \int_0^{1/\sqrt{3}} \frac{2}{(\sqrt{3}u)^2 + 1}\, du$$
$$\left[\begin{matrix}\text{Let } w = \sqrt{3}u \\ dw = \sqrt{3}\, du\end{matrix}\right] = \frac{2}{\sqrt{3}} \int_0^1 \frac{1}{w^2+1}\, dw = \frac{2}{\sqrt{3}} \arctan(w)\bigg|_0^1$$
$$= \frac{2}{\sqrt{3}}\left(\arctan(1) - \arctan(0)\right) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{4} - 0\right) = \frac{1}{\sqrt{3}}\pi$$

□

7. b)

$$\int_{2\sqrt{2}}^{4} \frac{1}{x\sqrt{x^2-4}}\,dx \quad \left[\begin{array}{c}\text{Let } x = 2\sec(u)\\ dx = 2\sec(u)\tan(u)\,du\end{array}\right]$$

$$= \int_{\text{arcsec}(\sqrt{2})}^{\text{arcsec}(2)} \frac{2\sec(u)\tan(u)}{2\sec(u)\sqrt{4\sec^2(u)-4}}\,du$$

$$= \frac{1}{2}\int_{\arccos(1/\sqrt{2})}^{\arccos(1/2)} \frac{\sec(u)\tan(u)}{\sec(u)\tan(u)}\,du = \frac{1}{2}\int_{\pi/4}^{\pi/3} du$$

$$= \frac{1}{2}u\Big|_{\pi/4}^{\pi/3} = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}\right)\pi = \frac{1}{24}\pi \qquad \square$$

7. c)

$$\int_0^{\pi/6} \cos^5(3x)\,dx \quad \left[\begin{array}{c}\text{Let } u = 3x\\ du = 3\,dx\end{array}\right] = \frac{1}{3}\int_0^{\pi/2} \cos^5(u)\,du$$

$$= \frac{1}{3}\int_0^{\pi/2} \left(1-\sin^2(u)\right)^2 \cos(u)\,dx \quad \left[\begin{array}{c}\text{Let } w = \sin(u)\\ dw = \cos(u)\,du\end{array}\right]$$

$$= \int_0^1 (1-w^2)^2\,dw = \int_0^1 \left[w^4 - 2w^2 + 1\right] dw$$

$$= \left(\frac{1}{5}w^5 - \frac{2}{3}w^3 + w\right)\Big|_0^1 = \left(\frac{1}{5} - \frac{2}{3} + 1\right) - 0$$

$$= \frac{3-10+15}{15} = \frac{8}{15} \qquad \square$$

7. d)

$$\frac{5x^2}{(x^2-4)(x^2+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x^2-4)}{(x^2-4)(x^2+1)}$$

$$= \frac{A(x^3+2x^2+x+2) + B(x^3-2x^2+x-2) + C(x^3-4x) + D(x^2-4)}{(x^2-4)(x^2+1)}$$

$$= \frac{(A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + (2A-2B-4D)}{(x^2-4)(x^2+1)}$$

$$\Rightarrow A+B+C=0,\ 2A-2B+D=5,\ A+B-4C=0,\ 2A-2B-4D=0$$

$$\Rightarrow A=1,\ B=-1,\ C=0,\ D=1$$

$$\int_0^1 \frac{5x^2}{(x^2-4)(x^2+1)}\,dx = \int_0^1 \left[\frac{1}{x-2} - \frac{1}{x+2} + \frac{1}{x^2+1}\right] dx$$

$$= \left(\log|x-2| - \log|x+2| + \arctan(x)\right)\Big|_0^1$$

$$= \left(\log(1) - \log(3) + \arctan(1)\right) - \left(\log(2) - \log(2) + \arctan(0)\right)$$

$$= \left(0 - \log(3) + \frac{\pi}{4}\right) - 0 = \frac{\pi}{4} - \log(3) \qquad \square$$