MAT 126: Problem Set 3 Solutions

1. (a) $$\int \sin^2(x)\,dx = \frac{1}{2}\int [1-\cos(2x)] = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C \qquad \square$$

1. (b) $$\int \cos^2(x)\sin^2(x)\,dx = \int \frac{1}{4}\left(1+\cos(2x)\right)\left(1-\cos(2x)\right)dx$$
$$= \int \left[\frac{1}{4} - \frac{1}{4}\cos^2(2x)\right]dx = \int \left[\frac{1}{4} - \frac{1}{8}(1+\cos(4x))\right]dx$$
$$= \int \left[\frac{3}{8} - \frac{1}{8}\cos(4x)\right]dx = \frac{3}{8}x - \frac{1}{32}\sin(4x) + C \qquad \square$$

1. (c) $$\int \cos^4(x)\,dx = \int \frac{1}{4}\left[1+\cos(2x)\right]^2\,dx$$
$$= \int \left[\frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right]dx$$
$$= \int \left[\frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8}\left(1+\cos(4x)\right)\right]dx$$
$$= \int \left[\frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right]dx$$
$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C \qquad \square$$

2. (a) $$\int \sin^2(x)\cos(x)\,dx \quad \begin{bmatrix}\text{Let } u = \sin(x) \\ du = \cos(x)\,dx\end{bmatrix} = \int u^2\,du$$
$$= \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C \qquad \square$$

2. (b) $$\int \sin^5(x)\,dx = \int \left(1-\cos^2(x)\right)^2 \sin(x)\,dx \quad \begin{bmatrix}\text{Let } u = \cos(x) \\ du = -\sin(x)\,dx\end{bmatrix}$$
$$= -\int (1-u^2)^2 du = -\int \left[1 - 2u^2 + u^4\right]du$$
$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$
$$= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C \qquad \square$$

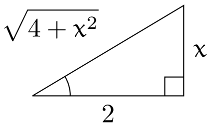
2. (c) $$\int \sin^4(x)\cos^7(x)\,dx$$
$$= \int \sin^4(x)\left(1-\sin^2(x)\right)^3\cos(x)\,dx \quad \begin{bmatrix}\text{Let } u = \sin(x) \\ du = \cos(x)\,dx\end{bmatrix}$$
$$= \int u^4\left(1-u^2\right)^3\,du = \int \left[u^4 - 3u^6 + 3u^8 - u^{10}\right]du$$
$$= \frac{1}{5}u^5 - \frac{3}{7}u^7 + \frac{1}{3}u^9 - \frac{1}{11}u^{11} + C$$
$$= \frac{1}{5}\sin^5 - \frac{3}{7}\sin^7 + \frac{1}{3}\sin^9 - \frac{1}{11}\sin^{11} + C \qquad \square$$

3. (a) $$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx \quad \left[\begin{array}{l}\text{Let } x = 2\tan(u)\\ dx = 2\sec^2(u)\,du\end{array}\right]$$

$$= \int \frac{2\sec^2(u)}{8\tan^2(u)\sqrt{1+\tan^2(u)}}\,du = \frac{1}{4}\int \frac{\sec(u)}{\tan^2(u)}\,du$$

$$= \frac{1}{4}\int \frac{1}{\cos(u)}\frac{\cos^2(u)}{\sin^2(u)}\,du = \frac{1}{4}\int \csc(u)\cot(u)\,du$$

$$= -\frac{1}{4}\csc(u) + C = -\frac{1}{4}\csc\left(\arctan(x/2)\right) + C$$

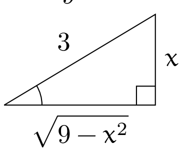


$$= -\frac{\sqrt{4+x^2}}{4x} + C$$ □

3. (b) $$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx \quad \left[\begin{array}{l}\text{Let } x = 3\sin(u)\\ dx = 3\cos(u)\,du\end{array}\right]$$

$$= \int \frac{3\cos(u)}{27\sin^2(u)\sqrt{1-\sin^2(u)}}\,du = \frac{1}{9}\int \csc^2(u)\,du$$

$$= -\frac{1}{9}\cot(u) + C = -\frac{1}{9}\cot\left(\arcsin(x/3)\right) + C$$



$$= -\frac{\sqrt{9-x^2}}{9x} + C$$

3. (c) $$\int \frac{x^2}{\sqrt{2x-x^2}}\,dx = \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx \quad \left[\begin{array}{c}\text{Let } x-1 = \sin(u)\\ dx = \cos(u)\,du\end{array}\right]$$

$$= \int \frac{(\sin(u)+1)^2\cos(u)}{\sqrt{1-\sin^2(u)}}\,du = \int \left[\sin^2(u) + 2\sin(u) + 1\right]du$$

$$= \int \left[-\frac{1}{2}\cos(2u) + 2\sin(u) + \frac{3}{2}\right]du$$

$$= -\frac{1}{4}\sin(2u) - 2\cos(u) + \frac{3}{2}u + C$$

$$= -\frac{1}{2}\sin(u)\sqrt{1-\sin^2(u)} - 2\sqrt{1-\sin^2(u)} + \frac{3}{2}u + C$$

$$= -\frac{1}{2}(x-1)\sqrt{2x-x^2} - 2\sqrt{2x-x^2} + \frac{3}{2}\arcsin(x-1) + C$$ □

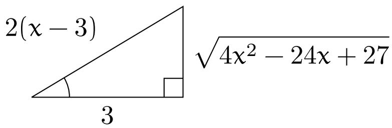
3. (d) $$\int \frac{1}{(4x^2-24x+27)^{3/2}}\,dx = \int \frac{1}{(4(x^2-6x)+27)^{3/2}}\,dx$$

$$= \int \frac{1}{(4\left((x-3)^2-9\right)+27)^{3/2}}\,dx = \int \frac{1}{(4(x-3)^2-9)^{3/2}}\,dx$$

$$= \int \frac{1}{4^{3/2}\left((x-3)^2-(3/2)^2\right)^{3/2}}\,dx \quad \left[\begin{array}{c} \text{Let } x-3 = \frac{3}{2}\sec(u) \\ dx = \frac{3}{2}\sec(u)\tan(u)\,du \end{array}\right]$$

$$= \int \frac{\frac{3}{2}\sec(u)\tan(u)}{4^{3/2}\left(\frac{3}{2}\right)^3(\sec^2(u)-1)^{3/2}}\,dx = \frac{1}{18}\int \frac{\sec(u)\tan(u)}{\tan^3(u)}\,du$$

$$= \frac{1}{18}\int \csc(u)\cot(u)\,du = -\frac{1}{18}\csc(u) + C$$



$$= -\frac{1}{9}\frac{x-3}{\sqrt{4x^2-24x+27}} + C$$ □

4. (a) $$\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx \quad [\text{Let } u = \tan(x/2)]$$

$$= \int \frac{1}{\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\,du = \int \frac{2}{1-u^2}\,du$$

$$= \int \left[\frac{1}{u+1} - \frac{1}{u-1}\right]du$$

$$= \log|\tan(x/2)+1| - \log|\tan(x/2)-1| + C$$ □

4. (b) $$\int \frac{5}{3\sin(x)+4\cos(x)}\,dx \quad [\text{Let } u = \tan(x/2)]$$

$$= \int \frac{5}{3\frac{2u}{1+u^2}+4\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\,du = \int \frac{10}{6u+4-4u^2}\,du$$

$$= \int \frac{-10}{(2u+1)(2u-4)}\,du = (*)$$

$$\frac{-10}{(2u+1)(2u-4)} = \frac{A}{2u+1} + \frac{B}{2u-4} = \frac{2(A+B)u-4A+B}{(2u+1)(2u-4)}$$

$$\Rightarrow A = 2, \quad B = -2$$

$$(*) = \int \left[\frac{2}{2u+1} - \frac{1}{u-2}\right]du = \log|2u+1| - \log|u-2| + C$$

$$= \log|2\tan(x/2)+1| - \log|\tan(x/2)-2| + C$$ □

4. (c) $\int \frac{1}{7\cos(x) - \sin(x) + 5}\, dx$ [Let $u = \tan(x/2)$]

$$= \int \frac{1}{7\frac{1-u^2}{1+u^2} - \frac{2u}{1+u^2} + 5} \frac{2}{1+u^2}\, du$$

$$= \int \frac{2}{7(1-u^2) - 2u + 5(1+u^2)}\, du = \int \frac{-1}{u^2+u-6}\, du = (*)$$

$$\frac{-1}{u^2+u-6} = \frac{-1}{(u+3)(u-2)} = \frac{A}{u+3} + \frac{B}{u-2}$$

$$= \frac{(A+B)u - 2A + 3B}{(u+3)(u-2)}$$

$$\Rightarrow A = \frac{1}{5}, \quad B = -\frac{1}{5}$$

$$(*) = \frac{1}{5}\int \left[\frac{1}{u+3} - \frac{1}{u-2}\right] du$$

$$= \frac{1}{5}\log|u+3| - \frac{1}{5}\log|u-2| + C$$

$$= \frac{1}{5}\log|\tan(x/2)+3| - \frac{1}{5}\log|\tan(x/2)-2| + C \qquad \square$$

5. (a)

$$F(x) = \int_1^x t^2 \sin(t)\, dt$$

$$F'(x) = x^2 \sin(x) \quad \textit{By FTC-II} \qquad \square$$

5. (b) $F(x) = \int_{-x}^{\log(x)\cos(x)} e^t \arctan(t)\, dt$

$$= \int_0^{\log(x)\cos(x)} e^t \arctan(t)\, dt - \int_0^{-x} e^t \arctan(t)\, dt$$

$$F'(x) = e^{\log(x)\cos(x)} \arctan(\log(x)\cos(x)) \left(\frac{\cos(x)}{x} - \log(x)\sin(x)\right)$$

$$- e^{-x}\arctan(-x)(-1) \quad \textit{By FTC-II}$$

$$= x^{\cos(x)} \arctan\left(\log(x)\cos(x)\right) \left(\frac{\cos(x)}{x} - \log(x)\sin(x)\right)$$

$$+ e^{-x}\arctan(-x) \qquad \square$$

5. (c)

$$F(x) = \int_2^{\int_0^x \arctan(t)\, dt} \tan(t)\, dt$$

$$F'(x) = \tan\left(\int_0^x \arctan(t)\, dt\right) \arctan(x) \quad \textit{By FTC-II} \qquad \square$$