MAT 126: Problem Set 1 Solutions

1. *Let* a, b, c *be positive real numbers greater than* 1. *Show that*

$$\log_a(bc)\log_b(ac)\log_c(ab) = \log_a(bc) + \log_b(ac) + \log_c(ab) + 2.$$

Solution: By logarithm identities we have:

$$\log_a(bc) = \frac{\log(bc)}{\log(a)} = \frac{\log(b) + \log(c)}{\log(a)}$$

Let $A = \log a$, $B = \log b$, and $C = \log c$. Then we have:

$$\begin{aligned}
&\log_a(bc)\log_b(ac)\log_c(ab) \\
=&\left(\frac{B+C}{A}\right)\left(\frac{A+C}{B}\right)\left(\frac{A+B}{C}\right) \\
=&\frac{1}{ABC}\left(AB + BC + AC + C^2\right)(A+B) \\
=&\frac{1}{ABC}\left(A^2B + ABC + A^2C + AC^2 + AB^2 + B^2C + ABC + BC^2\right) \\
=&\frac{A}{C} + 1 + \frac{A}{B} + \frac{C}{B} + \frac{B}{C} + \frac{B}{A} + 1 + \frac{C}{A} \\
=&\left(\frac{B+C}{A}\right) + \left(\frac{A+C}{B}\right) + \left(\frac{A+B}{C}\right) + 2 \\
=&\log_a(bc) + \log_b(ac) + \log_c(ab) + 2.
\end{aligned}$$

□

2. (a) *Show that if* a *and* h *are positive numbers with* $h < a^2$*, then*

$$\sqrt{a^2+h} - a < \frac{h}{2a} < a - \sqrt{a^2-h}.$$

Solution: By multiplying by the conjugate, we obtain:

$$\begin{aligned}
\sqrt{a^2+h} - a &= \left(\sqrt{a^2+h} - a\right)\frac{\sqrt{a^2+h}+a}{\sqrt{a^2+h}+a} \\
&= \frac{a^2+h-a^2}{\sqrt{a^2+h}+a} = \frac{h}{\sqrt{a^2+h}+a}.
\end{aligned}$$

Now, notice that, because we have $h > 0$, and since the function $f(x) = \sqrt{x}$ is increasing, we have $a^2 + h > a^2$ and thus $\sqrt{a^2+h} > \sqrt{a^2} = a$. Thus, $\sqrt{a^2+h} + a > 2a$. Since a dividing by a smaller number yields a bigger number, we have:

$$\sqrt{a^2+h} - a = \frac{h}{\sqrt{a^2+h}+a} < \frac{h}{2a}.$$

The other inequality is established similarly. We have:

$$\begin{aligned} a - \sqrt{a^2 - h} &= \left(a - \sqrt{a^2 - h}\right) \frac{a + \sqrt{a^2 - h}}{a + \sqrt{a^2 - h}} \\ &= \frac{a^2 - (a^2 - h)}{a + \sqrt{a^2 - h}} = \frac{h}{a + \sqrt{a^2 - h}}. \end{aligned}$$

As before, we have $a^2 - h < a^2$, so $a + \sqrt{a^2 - h} < \sqrt{a^2} + a = 2a$. We thus have:

$$\frac{h}{2a} < \frac{h}{a + \sqrt{a^2 - h}} = a - \sqrt{a^2 - h}. \qquad \square$$

2. (b) *Factor* $x^3 - y^3$ *and use this to show that if* a *and* h *are positive numbers with* $h < a^3$*, then*

$$\sqrt[3]{a^3 + h} - a < \frac{h}{3a^2} < a - \sqrt[3]{a^3 - h}.$$

Solution: We first begin by observing that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. This suggests that $x^2 + xy + y^2$ acts as a degree two analogue of the conjugate $x - y$. In particular, multiplying the top and bottom of $x - y$ by $x^2 + xy + y^2$ achieves the effect of cubing both summands.

With this insight, we manipulate the leftmost term in the inequality:

$$\begin{aligned} \sqrt[3]{a^3 + h} - a &= \left(\sqrt[3]{a^3 + h} - a\right) \frac{\left(\sqrt[3]{a^3 + h}\right)^2 + a\sqrt[3]{a^3 + h} + a^2}{\left(\sqrt[3]{a^3 + h}\right)^2 + a\sqrt[3]{a^3 + h} + a^2} \\ &= \frac{a^3 + h - a^3}{\left(\sqrt[3]{a^3 + h}\right)^2 + a\sqrt[3]{a^3 + h} + a^2} \\ &= \frac{h}{\left(\sqrt[3]{a^3 + h}\right)^2 + a\sqrt[3]{a^3 + h} + a^2} \end{aligned}$$

Subsequent to this, we once again have that $h > 0$ and that the cube root function is increasing, and so $\sqrt[3]{a^3 + h} > \sqrt[3]{a^3} = a$. From elementary properties of inequalities, we conclude that $\left(\sqrt[3]{a^3 + h}\right)^2 + a\sqrt[3]{a^3 + h} + a^2 > 3a^2$. We thus have

$$\sqrt[3]{a^3 + h} - a = \frac{h}{\left(\sqrt[3]{a^3 + h}\right)^2 + a\sqrt[3]{a^3 + h} + a^2} < \frac{h}{3a^2}.$$

For the other inequality we proceed similarly:

$$\begin{aligned}
a - \sqrt[3]{a^3 - h} &= \left(a - \sqrt[3]{a^3-h}\right) \frac{a^2 + a\sqrt[3]{a^3-h} + \left(\sqrt[3]{a^3-h}\right)^2}{a^2 + a\sqrt[3]{a^3-h} + \left(\sqrt[3]{a^3-h}\right)^2} \\
&= \frac{a^3 - (a^3 - h)}{a^2 + a\sqrt[3]{a^3-h} + \left(\sqrt[3]{a^3-h}\right)^2} \\
&= \frac{h}{a^2 + a\sqrt[3]{a^3-h} + \left(\sqrt[3]{a^3-h}\right)^2}.
\end{aligned}$$

We once gain have that $a^3 - h < a^3$ and thus $\sqrt[3]{a^3 - h} < \sqrt[3]{a^3} = a$. This yields $a^2 + a\sqrt[3]{a^3-h} + \left(\sqrt[3]{a^3-h}\right)^2 < 3a^2$. We thus have:

$$\frac{h}{3a^2} < \frac{h}{a^2 + a\sqrt[3]{a^3-h} + \left(\sqrt[3]{a^3-h}\right)^2} = a - \sqrt[3]{a^3 - h}. \qquad \square$$

2. (c) *Write* $\left|\left|83 - \sqrt{6891}\right| - \left|9 - \sqrt[3]{726}\right|\right|$ *without using absolute value signs.*

Solution: Notice that $6891 = 83^2 + 2$ and $726 = 9^3 - 3$. The innermost absolute value signs are thus trivial to remove, since $83 - \sqrt{83^2 + 2} < 0$ and $9 - \sqrt[3]{9^3 - 3} > 0$. We thus have:

$$\left|\left|83 - \sqrt{6891}\right| - \left|9 - \sqrt[3]{726}\right|\right| = \left|\left(\sqrt{83^2 + 2} - 83\right) - \left(9 - \sqrt[3]{9^3 - 3}\right)\right|.$$

By the inequalities from the previous parts, we have:

$$\sqrt{83^2 + 2} - 83 < \frac{2}{2 \cdot 83} = \frac{1}{83} < \frac{1}{81} = \frac{3}{3 \cdot 9^2} < 9 - \sqrt[3]{9^3 - 3}.$$

We are thus currently subtracting a bigger number from a smaller one, and so the expression in the absolute value bars is negative. Hence we conclude:

$$\left|\left|83 - \sqrt{6891}\right| - \left|9 - \sqrt[3]{726}\right|\right| = \left(9 - \sqrt[3]{726}\right) - \left(\sqrt{6891} - 83\right).$$

At this point we are done, but if you're feeling particularly keen on doing arithmetic, then you can write this as $92 - \sqrt[3]{726} - \sqrt{6891}$. $\square$

3. *Let* $r \in \mathbb{R}$ *and consider the sequence* $x_n = r^n$. *How does this sequence behave for different values of* r*? For which* r *does* $\lim_n r^n$ *exist?*

Solution: If $|r| < 1$, then each subsequent term of the sequence r^n multiplies each previous term by a number less than 1 in absolute value. These terms are thus getting smaller. This alone does not "*prove*" that the sequence converges to 0, but the Greeks knew that if you kept multiplying by numbers uniformly bounded away from 1, then you would eventually have something arbitrarily small – this is the *Exhaustion Lemma* of Eudoxus.

In this question, **you were not asked to establish anything formally**, but I can present several quick semi-formal arguments for this fact:

a) If we accept the formula for the sum of the infinite sum of the geometric series $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, then we see in the case that $0 \leqslant r < 1$ that we have a sum of infinitely many non-negative terms that converges to a finite number. If these terms were not eventually arbitrarily small, then this could not happen. The terms in the case of $-1 < r < 0$ differs only by a sign, and so we can conclude that they are small from the positive case.

b) Again, in the case $0 \leqslant r < 1$, we have that r^n is a bounded monotonically decreasing sequence. A theorem about sequences then tells us that it must converge to some value L. If $L \neq 0$, then we can we have that the sequence is eventually at most $L\frac{1-r}{1+r}$ away from L. However, if $|L - r^n| < L\frac{1-r}{1+r}$, then $r^n < L + L\frac{1-r}{1+r} = L\frac{2}{1+r}$ and thus $r^{n+1} < L\frac{2r}{1+r}$. Now $r^{n+1} - L < L\frac{2r}{1+r} - L = L\frac{1-r}{1+r}$. But this says that $|L - r^{n+1}| \not< L\frac{1-r}{1+r}$, so this inequality cannot hold *eventually*. This is a contradiction, so we conclude $L = 0$. The case for $-1 < r < 0$ follows since the terms only differ by a sign.

Now, for $r = -1$, we see that the sequence alternates between -1 and 1 and thus does not converge to any specific value. Meanwhile, for $r = 1$, the sequence is always 1 and thus converges to 1. Finally, if $|r| > 1$, then $|r^n|$ grows like an exponential function. When $r > 1$, the sequence diverges to ∞ (which is not a number, so the sequence is not convergent). When $r < -1$, the sequence blows up, but does so with alternating signs (which means that we can't even say $\lim_n r^n = -\infty$, so it fails to converge in an even worse sense). □

4. *Show that* $\lim_n \sqrt[n]{2^n + 5^n}$ *exists and find its value.*

Solution: This follows by heavy use of limit laws:

$$\lim_n \sqrt[n]{2^n + 5^n} = \lim_n 5 \sqrt[n]{\left(\frac{2}{5}\right)^n + 1} = 5 \sqrt[n]{\lim_n \left(\left(\frac{2}{5}\right)^n + 1\right)}$$
$$= 5 \sqrt[n]{\lim_n \left(\frac{2}{5}\right)^n + \lim_n 1} = 5 \sqrt[n]{0 + 1} = 5.$$

Bringing the limit inside the radical is allowed since it was said in class that you may assume continuity of the function $f(x) = \sqrt[n]{x}$. Further, by the previous problem, we know that $\lim_n \left(\frac{2}{5}\right)^n = 0$. □

5. *Show that* $f(x) = x^2$ *is continuous at every* $a \in \mathbb{R}$.

Solution: Let $a \in \mathbb{R}$. We will show that $\lim_{x\to a} f(x) = f(a)$. Towards this end, consider:

$$|x^2 - a^2| = |x + a| \cdot |x - a|.$$

If $|x - a| < 1$, then $|x + a| < 2|a| + 1$, and so $|x^2 - a^2| < (2|a| + 1) \cdot |x - a|$.

Now, let $A \in \mathcal{O}(a^2)$. We then have that there is some $a^2 \in (x, y) \subseteq A$. Let $\epsilon = \min\{a^2 - x, y - a^2\}$. Then $a^2 \in (a^2 - \epsilon, a^2 + \epsilon) \subseteq (x, y) \subseteq A$. Now, let $\delta = \min\left\{\frac{\epsilon}{2|a|+1}, 1\right\}$, and $B = (a - \delta, a + \delta)$. If $x \in B$, then $|x - a| < a$, and so

$$|x^2 - a^2| < (2|a| + 1) \cdot |x - a| < (2|a| + 1)\,\frac{\epsilon}{2|a| + 1} = \epsilon.$$

Hence $x^2 \in (a^2 - \epsilon, a^2 + \epsilon)$, and so $f(x) \in A$. Thus $f[B] \subseteq A$. Hence $\lim_{x\to a} f(x) = f(a)$, and so f is continuous. $\square$

6. *Using Riemann sums, show that the function* $f(x) = x^3$ *is integrable on the interval* $[0, 1]$ *and compute* $\int_0^1 f(x)\,dx$.

Solution: Since f is increasing on the interval $[0, 1]$, we have, for any $X = [x, y] \subseteq [0, 1]$, that $\inf_X f = f(x)$ and $\sup_X f = f(y)$. Now, let P_n be the n-th uniform partition, such that $P_n \equiv \left(0 < \frac{1}{n} < \ldots < \frac{n-1}{n} < \frac{n}{n}\right)$. We then have:

$$\begin{aligned}
U(f, P_n) &= \sum_{k=1}^{n} \left(\frac{k}{n} - \frac{k-1}{n}\right) \sup_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} f \\
&= \sum_{k=1}^{n} \frac{1}{n} \left(\frac{k}{n}\right)^3 \\
&= \frac{1}{n^4} \left(\sum_{k=1}^{n} k^3\right) \\
&= \frac{1}{n^4} \frac{n^2(n+1)^2}{4} \\
&= \frac{\left(1 + \frac{1}{n}\right)^2}{4}
\end{aligned}$$

Using the sum of cubes formula given in the notes. We similarly have:

$$\begin{aligned} L(f, P_n) &= \sum_{k=1}^{n} \left(\frac{k}{n} - \frac{k-1}{n} \right) \inf_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} f \\ &= \sum_{k=1}^{n} \frac{1}{n} \left(\frac{k-1}{n} \right)^3 \\ &= \frac{1}{n^4} \left(\sum_{k=1}^{n-1} k^3 \right) \\ &= \frac{1}{n^4} \frac{(n-1)^2 n^2}{4} \\ &= \frac{\left(1 - \frac{1}{n}\right)^2}{4} \end{aligned}$$

Using limit laws and the fact that $\lim_n \frac{1}{n} = 0$, we have

$$\lim_n U(f, P_n) = \lim_n L(f, P_n) = \frac{1}{4}.$$

By the definition of an integral, this tells us that $\int_0^1 f(x)\,dx = \frac{1}{4}$. $\square$

7. *Using Riemann sums, show that the function*

$$f(x) = \begin{cases} 0 & x < 1/2 \\ 1 & x \geqslant 1/2 \end{cases}$$

is integrable on the interval $[0, 1]$ *and compute* $\int_0^1 f(x)\,dx$.

Solution: Consider $P_n = \left(0 < \frac{1}{2} - \frac{1}{n} < \frac{1}{2} + \frac{1}{n} < 1\right)$ for $n \geqslant 2$. Notice that f is uniformly 0 on $\left[0, \frac{1}{2} - \frac{1}{n}\right]$ and uniformly 1 on $\left[\frac{1}{2} + \frac{1}{n}, 1\right]$. Meanwhile, on the interval $\left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right]$, f takes on both of the values 0 and 1. We thus have:

$$\begin{aligned} L(f, P_n) = &\left(\left(\frac{1}{2} - \frac{1}{n} \right) - 0 \right) \cdot 0 \\ &+ \left(\left(\frac{1}{2} + \frac{1}{n} \right) - \left(\frac{1}{2} - \frac{1}{n} \right) \right) \cdot 0 \\ &+ \left(1 - \left(\frac{1}{2} + \frac{1}{n} \right) \right) \cdot 1 \\ = &\frac{1}{2} - \frac{1}{n}. \end{aligned}$$

Similarly for upper sums:

$$\begin{aligned} U(f, P_n) = &\left(\left(\frac{1}{2} - \frac{1}{n}\right) - 0\right) \cdot 0 \\ &+ \left(\left(\frac{1}{2} + \frac{1}{n}\right) - \left(\frac{1}{2} - \frac{1}{n}\right)\right) \cdot 1 \\ &+ \left(1 - \left(\frac{1}{2} + \frac{1}{n}\right)\right) \cdot 1 \\ =&\frac{1}{2} + \frac{1}{n}. \end{aligned}$$

Using limit laws and the fact that $\lim_n \frac{1}{n} = 0$, we have

$$\lim_n U(f, P_n) = \lim_n L(f, P_n) = \frac{1}{2}.$$

By the definition of an integral, this tells us that $\int_0^1 f(x)\, dx = \frac{1}{2}$. □

8. *Show that* $|3 \sin \theta + 4 \cos \theta| \leqslant 5$*. When does equality hold?*

Solution: Let $\alpha = \arctan\left(\frac{3}{4}\right)$. Then $\sin(\alpha) = \frac{3}{5}$ and $\cos(\alpha) = \frac{4}{5}$. [This may be verified by checking that $\frac{\sin(\alpha)}{\cos(\alpha)} = \frac{3}{4}$ and that the point $\left(\frac{4}{5}, \frac{3}{5}\right)$ lies on the unit circle.] We then have:

$$3 \sin \theta + 4 \cos \theta = 5\left(\sin(\alpha) \sin(\theta) + \cos(\alpha) \cos(\theta)\right).$$

By the angle subtraction formula for $\cos$, this is equal to $5 \cos(\theta - \alpha)$. Hence:

$$|3 \sin \theta + 4 \cos \theta| = 5|\cos(\theta - \alpha)| \leqslant 5,$$

since $\cos$ ranges in values from -1 to 1.

We have equality precisely when $\cos(\theta - \alpha) = \pm 1$. This occurs when $\theta - \alpha = 2k\pi$ or when $\theta - \alpha = \pi + 2k\pi$. In other words, we have equality precisely at the points:

$$\left\{\arctan\left(\frac{3}{4}\right) + 2k\pi : k \in \mathbb{Z}\right\} \cup \left\{\arctan\left(\frac{3}{4}\right) + 2(k+1)\pi : k \in \mathbb{Z}\right\}.$$ □