MAT 126: Problem Set 1

1. Let a, b, c be positive real numbers greater than 1. Show that

$$\log_a(bc)\log_b(ac)\log_c(ab) = \log_a(bc) + \log_b(ac) + \log_c(ab) + 2.$$

Hint: Express everything in terms of $A = \log a$, $B = \log b$, and $C = \log c$.

2. (a) Show that if a and h are positive numbers with $h < a^2$, then

$$\sqrt{a^2+h} - a < \frac{h}{2a} < a - \sqrt{a^2-h}.$$

(b) Factor $x^3 - y^3$ and use this to show that if a and h are positive numbers with $h < a^3$, then

$$\sqrt[3]{a^3+h} - a < \frac{h}{3a^2} < a - \sqrt[3]{a^3-h}.$$

(c) Write $\left|\left|83 - \sqrt{6891}\right| - \left|9 - \sqrt[3]{726}\right|\right|$ without using absolute value signs. *Use (a) and (b), but do not use a calculator.*

3. Let $r \in \mathbb{R}$ and consider the sequence $x_n = r^n$. How does this sequence behave for different values of r? For which r does $\lim_n r^n$ exist?

4. Show that $\lim_n \sqrt[n]{2^n + 5^n}$ exists and find its value.

5. Show that $f(x) = x^2$ is continuous at every $a \in \mathbb{R}$.
Hint: Recall that f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

6. Using Riemann sums, show that the function $f(x) = x^3$ is integrable on the interval $[0, 1]$ and compute $\int_0^1 f(x)\,dx$.

7. Using Riemann sums, show that the function

$$f(x) = \begin{cases} 0 & x < 1/2 \\ 1 & x \geqslant 1/2 \end{cases}$$

is integrable on the interval $[0, 1]$ and compute $\int_0^1 f(x)\,dx$.
Hint: Use $P_n \equiv (0 < \frac{1}{2} - \frac{1}{n} < \frac{1}{2} + \frac{1}{n} < 1)$.

8. (**Bonus**) Show that $|3\sin\theta + 4\cos\theta| \leqslant 5$. When does equality hold?
Use trig identities, not calculus, to do this exercise.
Hint: There is an angle α with $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.