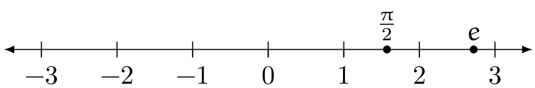
1.1 A Bit of Set Theoretic Language

In mathematics, related things are often collected into sets. If something is "*a thing*", then there is usually some set classifying all things of that genus. Some examples of this are the various notions that we have for numbers:

1. The **natural numbers** $\mathbb{N} = \{0, 1, 2, \ldots\}$.
2. The **integers** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
3. The **rational numbers** $\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \right\}$.
4. The **real numbers** $\mathbb{R}$, whose elements are often pictured as points along "*the number line*":



Once we accept that real numbers *are things*, then another kind of thing that we can consider is ordered pairs of real numbers, such as $(1, 4)$. The set of all such pairs forms the **cartesian plane**. You are all used to plotting ordered pairs of numbers on a plane, reading the components of the pair as *coordinates*.

In general, if X and Y are sets, then we have a set $X \times Y$ of ordered pairs with first component in X and second component in Y. We write:

$$X \times Y = \{(a, b) \mid a \in X,\ b \in Y\}.$$

This set is known as the **cartesian product** of X and Y.

If all of the elements of A are elements of X, then we say that A is a **subset** of X, and write $A \subseteq X$. For example, we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

If X is a set, then a subset of X is *a thing*, and thus there should be a set of all subsets of X. This is known as the **power set** of X. We write:

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}.$$

Combining these two notions, we can consider $\mathcal{P}(X \times X)$, the set of sets of ordered pairs with components lying in X. An element of $\mathcal{P}(X \times X)$ is known as a **relation** on X. (A relation is just a collection of pairs.) For example, we may consider:

$$\left\{(x, y) \mid x^2 + y^2 = 1\right\} \subseteq \mathbb{R} \times \mathbb{R},$$

this is the set of points in the cartesian plane that lie on the unit circle. Further, if $f : \mathbb{R} \to \mathbb{R}$ is any function, then we have

$$\{(x, f(x)) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R},$$

which is the **graph** of f.

1.2 Orderings

Key takeaway: $\mathcal{O}(x)$, ordered by reverse-inclusion, is a directed set.

One interesting relation on $\mathbb{R}$ is $\leqslant$. Thought of as a set of pairs, this is:

$$\{(x, y) \mid x \leqslant y\} \subseteq \mathbb{R} \times \mathbb{R}.$$

In other words, the relation $\leqslant$ can be defined by specifying all pairs (x, y) such that $x \leqslant y$. We take this relation as understood. For example, $2 \leqslant 3$.

If $a, b \in \mathbb{R}$, then we may consider several kinds of **intervals**:

$$\begin{aligned} (a, b) &= \{x \mid a < x < b\} \\ [a, b] &= \{x \mid a \leqslant x \leqslant b\} \\ (a, b] &= \{x \mid a < x \leqslant b\} \\ [a, \infty) &= \{x \mid a \leqslant x\}. \end{aligned}$$

Note that the notation (a, b) is used for two *very different* things: the ordered pair (a, b) and the interval (a, b). You have to infer which of these two meanings is intended from context.

We can list several properties of the relation $\leqslant$:

1. **Reflexively**: For all x, $x \leqslant x$.
2. **Transitivity**: If $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.
3. **Anti-symmetry**: If $x \leqslant y$ and $y \leqslant x$, then $x = y$.
4. **Totality**: For any x and y, either $x \leqslant y$ or $y \leqslant x$.

If X is any set and $R \subseteq X \times X$ is a relation satisfying the four properties above, then we say that R is a **total order** on X. For example, there are standard total orders, all denoted by $\leqslant$, on the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$.

On the other hand, if $R \subseteq X \times X$ is assumed only to satisfy the first three properties, then we say that R is a **partial order** on X. For example the relation $\subseteq$ on $\mathcal{P}(X)$ is a partial order.

In this course, we often want to order $\mathcal{P}(X)$ by **reverse-inclusion**. If $A, B \in \mathcal{P}(X)$, then we say $A \leqslant B$ if $B \subseteq A$. In other words, subsets are "*bigger*" than their parent sets.

Notice that if $A, B \in \mathcal{P}(X)$, then $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so $A \leqslant A \cap B$ and $B \leqslant A \cap B$. So, while two sets A and B may be incomparable, there must exist some subset of both sets, for example, their intersection.

If $\leqslant$ is a partial order on X, then we say that $\leqslant$ is **directed** if for every $x, y \in X$, there is some $z \in X$ with $x \leqslant z$ and $y \leqslant z$. Roughly, this means that all paths up the order unify. Directed orders look like they're going somewhere.

As a directed order, $\leqslant$ on $\mathcal{P}(X)$ is rather boring, since for any $A \subseteq X$, we have $\varnothing \subseteq A$, and hence $A \leqslant \varnothing$. In other words, $\varnothing$ is the **maximum** element of $\mathcal{P}(X)$.

To live is to yearn and strive, yet to never reach completion. We would like to consider directed partial orders with this property. As we will see, much of the theory of calculus comes down to such directed sets.

Definition Let $x \in \mathbb{R}$. We say that a set $A \subseteq \mathbb{R}$ is a **neighbourhood of** x if there is some interval (a, b) with $x \in (a, b) \subseteq A$. (In other words, if x is in A, and if x further has some *wiggle room*.) We write $\mathcal{O}(x)$ for the set of neighbourhoods of x. ■

Since $\mathcal{O}(x) \subseteq \mathcal{P}(\mathbb{R})$, $\mathcal{O}(x)$ inherits the order $\leqslant$ of reverse-inclusion. The subtlety of this ordered set is that if $A \in \mathcal{O}(x)$, then the amount of wiggle room that x has in A can be made smaller, but never eliminated. For example, in $\mathcal{O}(0)$, we have:

$$\left[-\frac{1}{2}, \frac{1}{2}\right] \leqslant \left[-\frac{1}{3}, \frac{1}{3}\right] \leqslant \left[-\frac{1}{4}, \frac{1}{4}\right] \leqslant \left[-\frac{1}{5}, \frac{1}{5}\right] \leqslant \dots$$

However $\varnothing \notin \mathcal{O}(0)$ because $0 \notin \varnothing$, also $\{0\} \notin \mathcal{O}(0)$ because 0 has no wiggle room in $\{0\}$. (These two would be the naïve candidates for a maximal element of $\mathcal{O}(0)$.)

Here is a key property: Suppose that $A, B \in \mathcal{O}(x)$, then there exist intervals such that $x \in (a, b) \subseteq A$ and $x \in (c, d) \subseteq B$. Then $a < x$ and $c < x$, so $\max(a, c) < x$. Similarly $x < \min(b, d)$. Thus:

$$x \in (\max(a, c), \min(b, d)) \subseteq A \cap B$$

In other words, $A \cap B \in \mathcal{O}(x)$. That is, the intersection of two neighbourhoods of x is a neighbourhood of x. Then, as we observed above, $A \leqslant A \cap B$ and $B \leqslant A \cap B$. <u>This shows that $\mathcal{O}(x)$ is a directed set!</u>

1.3 Limits

Key takeaway: "*Eventually*" is parametrised by a directed set. "*Arbitrarilly close*" is parematrised by the neighbourhoods of a point.

The most basic kind of limit (you will see a lot more of these in *Calculus C*) is taking a limit of a sequence. The quickest definition of a sequence is to say that a **sequence** is a function $\mathbb{N} \to \mathbb{R}$. In other words, a sequence is an assignment to each natural number of a real number.

We can express sequences like: $(0, 0, 0, 0, \ldots)$ or $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$; here we rely on the reader to spot a pattern in the terms and extend its definition infinitely to the right. We could also define these same sequences as $a_n = 0$ and $b_n = \frac{1}{n+1}$; here we give a general formula for the n-th term.

For these sequences, a_n is always 0, while b_n gets arbitrarily close to 0. We should have that both sequences *tend to* 0, and we should be able to write $\lim_n a_n = \lim_n b_n = 0$. Informally, if x_n is any sequence, then we say that $\lim_n x_n = l$ if x_n is "*eventually*" "*arbitrarily close*" to l. How do we make this precise?

The notion of "*eventually*" makes sense in any directed set. If $(X, \leqslant)$ is a directed set, set then we say that some condition parametrised by X holds eventually if there is some $x \in P$ such that the condition is true of all $y \in X$ with $x \leqslant y$. The word "*eventually*" is suitable for this context as moving upwards from any element will eventually lead you to some element larger than x. In our particular case, $\mathbb{N}$ is totally order, and hence is a directed set. Thus to say that some condition holds eventually for the sequence x_n would be to say that there is some $N \in \mathbb{N}$ such that the condition holds for all x_n with $n \geqslant N$.

Now, what about "*arbitrarily close*"? Well, you can think of $\mathcal{O}(l)$ as paremetrising all relative degrees of closeness to l. Your requirement of closeness can't stipulate that something is exactly equal to l; you always have to include some wiggle room, and so considering neighbourhoods is appropriate.

Putting this together, we get that x_n converges to l if x_n eventually lies in any neighbourhood of l. Or formally:

Definition Let x_n be a sequence and let $l \in \mathbb{R}$. Then we say x_n **converges to** l, and write $\lim_n x_n = l$, if for any $A \in \mathcal{O}(l)$, there exists some $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $x_n \in A$. ■

We will now work through an example of how to carefully show that a sequence has a certain limit. **Pay careful attention to examples**; they explain how you actually use the material to solve HW problems!

Example Let $x_n = \frac{1}{n+1}$. Show that $\lim_n x_n = 0$.

Solution Let $A \in \mathcal{O}(0)$. By the definition of a neighbourhood, we have some $a, b \in \mathbb{R}$ such that $0 \in (a, b) \subseteq A$. Since $0 \in (a, b)$, we have $a < 0 < b$. Now, since $b > 0$, $\frac{1}{b}$ is some strictly positive number. Let N be the smallest integer greater than or equal to $\frac{1}{b}$, i.e. $N = \left\lceil \frac{1}{b} \right\rceil$.

Whenever $n \geqslant N$, we have $\frac{1}{b} \leqslant N \leqslant n < n+1$, and so $\frac{1}{n+1} < b$. Further, we have $a < 0 < \frac{1}{n+1}$. Thus we have $a < x_n < b$, and so $x_n \in (a, b) \subseteq A$. Hence, whenever $n \geqslant N$, $x_n \in A$. □

The type of limits that we saw in *Calculus A* were not of this form; we instead spoke of taking the limit of a function as its argument approached some point. Once again, the informal definition of $\lim_{x \to a} f(x) = l$ is that when x is "*sufficiently close*" to a, $f(x)$ is "*arbitrarily close*" to l.

We already know that closeness is parametrised by the neighbourhoods of a point. The phrase "*sufficiently close*" means "*when x is close enough*" and the word "*enough*" suggests "*eventually*". This suggests that, as opposed to using the natural numbers as our indexing set, we should consider some mathematical expression parametrised by the neighbourhoods of a. $\mathcal{O}(a)$ forms a directed set, and so the notion of "*eventually*" makes sense.

If $B \in \mathcal{O}(a)$, then we can consider the values that f takes on B. The **image** of B under f is defined as:

$$f[B] = \{f(x) \mid x \in B\}.$$

The informal definition of limits of a function suggested that we require that when B is "*small enough*", then all of the values that f takes on B be arbitrarily close to l. Well, arbitrary closeness says that we can make this happen for any choice of neighbourhood $A \in \mathcal{O}(l)$. Being within the precision stipulated by A just means that $f[B] \subseteq A$.

Finally, notice that if $B' \subseteq B$, then $f[B'] \subseteq f[B]$. Thus if $f[B] \subseteq A$, then $f[B'] \subseteq A$ for any $B' \geqslant B$. So, to show that $f[B]$ is eventually contained in A requires finding just one B with this property. Wrapping this up in a formal definition, we obtain:

Definition Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a, l \in \mathbb{R}$. We say that $f(x)$ **converges to l as x tends to a**, and write $\lim_{x \to a} f(x) = l$, if for any $A \in \mathcal{O}(l)$, there is some $B \in \mathcal{O}(a)$ such that $f[B] \subseteq A$. ■

Example Let $f(x) = x^2$. Then $\lim_{x \to 0} f(x) = 0$.

Solution Let $A \in \mathcal{O}(0)$. By the definition of a neighbourhood, we have some $a, b \in \mathbb{R}$ such that $0 \in (a, b) \subseteq A$. Since $0 \in (a, b)$, we have

$a < 0 < b$. Now, since $b > 0$, we may take its square root, so let $B = \left(-\sqrt{b}, \sqrt{b}\right)$. B is clearly a neighbourhood of 0.

If $x \in B$, then $|x| < \sqrt{b}$, and so $x^2 < \sqrt{b}^2 = b$. We also have $0 \leqslant x^2$. Thus $a < 0 \leqslant x^2 < b$, and so $f(x) \in A$, hence $f[A] \subseteq B$. $\square$

Notice that the image of any set under squaring never contains negative numbers. This shows that $f[B]$ will not in general be a neighbourhood of $f(a)$, since, in this example, 0 has no wiggle room downwards.

A key use of limits is to define derivatives:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

However, we have an issue with this expression and our prior definition, since the difference quotient is not defined when $x = a$ or $h = 0$. We thus do not have a function $g : \mathbb{R} \to \mathbb{R}$. Rather, the difference quotient is a *partial function* with domain strictly smaller than $\mathbb{R}$. In general, if the domain of f is $X \subseteq \mathbb{R}$, we write $f : X \to \mathbb{R}$.

When taking limits of partial functions, we would like for the function to be defined on at least some points arbitrarily close to a, as otherwise, $f[B]$ would eventually be empty, and then we could say that the limit of f is any value whatsoever. In general, if $X \subseteq \mathbb{R}$ is the domain of f, then we say that a is a **cluster point** of X if for any $B \in \mathcal{O}(a)$, the intersection $B \cap X$ is non-empty. With this notion, we can generalise out definition of limits to partial functions:

Definition Let $f : X \to \mathbb{R}$ be a function and let $a, l \in \mathbb{R}$ such that a is a cluster point of X. We say that $f(x)$ **converges to** l **as** x **tends to** a, and write $\lim_{x \to a} f(x) = l$, if for any $A \in \mathcal{O}(l)$, there is some $B \in \mathcal{O}(a)$ such that $f[B \cap X] \subseteq A$. $\blacksquare$

This powered-up version of limits lets us define derivatives, and from this point, we get all of *Calculus A*.

1.4 sup **and** inf

Before we move on to integration, we will study a very special property of the order on $\mathbb{R}$.

Definition Let $A \subseteq \mathbb{R}$. An **upper bound** of A is a number $M \in \mathbb{R}$ such that $x \leqslant M$ for all $x \in A$. If A has an upper bound, then we say that A is **bounded above**. $\blacksquare$

For example, the set $[0, \infty)$ is not bounded above, while the set $[0, 1]$ has 2 as an upper bound. Of course, 1 is also an upper bound on $[0, 1]$, and in some sense, 1 is the *best* upper bound that one can give. We can make this precise:

Definition Let $A \subseteq \mathbb{R}$ be bounded above. A **least upper bound** of A is an upper bound M of A such that for any other upper bound M' of A, we have $M \leqslant M'$. If a least upper bound exists, then it is unique, and we denote its value by $\sup A$, also called the **supremum** of A. ■

As another example, $\sup [0, \pi) = \pi$. On the other hand, if we instead think about the order on $\mathbb{Q}$ and consider the set $A = [0, \pi) \cap \mathbb{Q}$ of rational numbers, then A is bounded above (e.g. by 4), but A does not have a least upper bound in $\mathbb{Q}$ since $\pi \notin \mathbb{Q}$. The difference between $\mathbb{R}$ and $\mathbb{Q}$ as totally ordered sets, in some sense, is that $\mathbb{Q}$ has holes, while $\mathbb{R}$ is *complete*.

Indeed, in a sense that may be made formal, $\mathbb{R}$ may be uniquely characterised as being a totally ordered set with addition, subtraction, multiplication, and division, and such that the following property holds:

Axiom: Let $A \subseteq \mathbb{R}$ be a non-empty set of real numbers such that A is bounded above. Then $\sup A$ exists. ■

Let's introduce one piece of notation: Suppose the f is a function defined on $X \subseteq \mathbb{R}$ with X non-empty. We say that f *is bounded above on* X if $f[X]$ is bounded above. In this case, we denote $\sup f[X]$ by $\sup_X f$.

We now state some elementary properties of $\sup$:

1. Suppose that $A \subseteq B \subseteq \mathbb{R}$ such that A is non-empty and such that B is bounded above. Then $\sup A \leqslant \sup B$.

2. Suppose that f, g are functions defined on X, with X non-empty, and such that f and g are both bounded above on X. Then $f+g$ is bounded above on X, and $\sup_x (f+g) \leqslant \sup_x f + \sup_X g$.

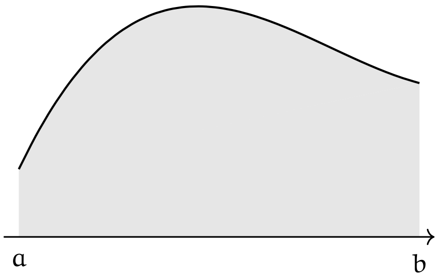
By modifying the definitions above, we obtain the notion of a **lower bound** and **greatest lower bound** of some set $A \subseteq \mathbb{R}$. If the greatest lower bound of A exists, then we denote it by $\inf A$ and call it the **infimum** of A. If A is a non-empty set of real numbers such that A is bounded below, then $\inf A$ exists. Similarly, analogous of the above two properties hold of $\inf$.

Finally, we say that a set A is bounded if A is bounded both above and below. We may similarly speak of a function f being bounded on X.

1.5 Integration

Note: In this section we assume that $L = [a, b]$ is an interval with $a < b$.

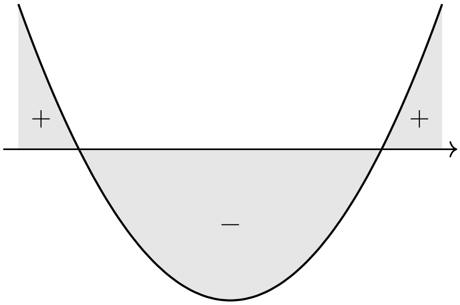
In the picture below, we have shaded in the area lying between the graph of some function f and the x-axis, along an interval $[a, b]$.



The word *area* has two different uses in mathematics. In the paragraph above, we have used it to refer to some *region* of the plane. Its other meaning refers to a numerical value expressing exactly how much "*area*" a region occupies.

It is very straightforward to give a criterion for when a point (x, y) belongs to the shaded region of the plane, viz., assuming that $f \geqslant 0$ on $[a, b]$, this occurs exactly when $a \leqslant x \leqslant b$ and $0 \leqslant y \leqslant f(x)$. What is far less straightforward is to quantify the *area* of this region. As it turns out, this is not just an idle question, but rather hits at the heart of calculus and is central to most of its wide reaching applications.

The first step in resolving this problem is to explain what it even means to take the area of a region. (As you may expect, this is a rather subtle concept.) In fact, we will restrict the scope of the problem to regions of a very special form – those defined by the regions lying between the graph of a function and the x-axis. If we succeed in giving a value to such a region, then we will refer to that value as the **area under the curve**.



We should provide one clarification to this notion. Suppose that, as in the diagram above, f is not necessarily assumed to lie over the x-axis. Our notion of *area under the curve* will refer to a *signed area* given by the difference of the areas of the regions lying over and under the x-axis. (In the picture above, it looks like there's more area below the x-axis than above it, so the area under the curve, provided that we succede in defining it, will be negative.)

With this clarification, we now understand the goal of this section of the notes: given a function f defined on $L = [a, b]$ to define the notion of the *area under the curve*. Towards this end, we will consider the notion of a *partition* of L.

Definition A **partition** of L is a finite subset $P \subseteq L$ such that P contains the endpoints a and b. We usually write a partition by ordering its points. For example, if P has n points, then we would write:

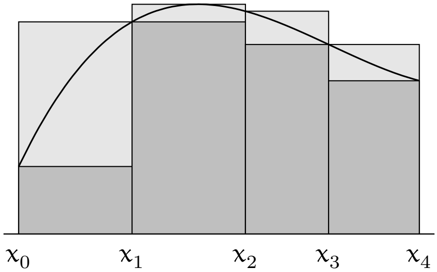
$$P \equiv (a = x_0 < \ldots < x_n = b).$$

A partition is to be thought of as corresponding to a subdivision of L into pieces joind by their endpoints: $L = [x_0, x_1] \cup \ldots \cup [x_{n-1}, x_n]$. The set of partitions of L is denoted by Δ_L. ∎

Since a partition is a set of points, the set of partitions may be ordered by containment. Ln this case, we will actually want to order them by inclusion, not reverse-inclusion. Thus, if $P, Q \in \Delta_L$, then we say $P \leqslant Q$ if $P \subseteq Q$. In other words, the partitions further up in the ordering subdivide L into finer pieces. If $P \leqslant Q$, we say that Q is **finer** than (or a **refinement** of) P.

If $P, Q \in \Delta_L$, then the union of their sets of division points $P \cup Q$ is a finite set containing the endpoints of L, and hence $P \cup Q$ is a partition of L. We have that $P \leqslant P \cup Q$ and $Q \leqslant P \cup Q$, so $P \cup Q$ is a **common refinement** of P and Q. This construction shows that Δ_L is a directed set.

Suppose that we are given a partition $P \equiv (x_0 < x_1 < x_2 < x_3 < x_4)$ of the interval on which f is defined in the first diagram above.



Since the function pictured above is bounded, we may consider the quantities $\sup_{[x_{i-1},x_i]} f$ and $\inf_{[x_{i-1},x_i]} f$ over any of the intervals defined by the partition. By definition, these are the best bounds that we can obtain for the values that f takes on $[x_{i-1}, x_i]$. Over each interval, we may consider forming rectangles with base $[x_{i-1}, x_i]$ and height given by either the supremum or infimum of f over the interval. We will call these the upper and lower rectangles, respectively. The union of all of the upper, or of all of the lower, rectangles may be thought of as an *approximation* to the region bound by the curve.

One very reasonable assertion that we can make about area is that the area of a rectangle is the product of its side lengths. Further, if we take a union of rectangles that intersect only along their boundary, the area of the total region is the sum of the areas of the individual rectangles. Thus, the interest in considering the approximate regions formed by the upper or lower rectangles, as above, is that these approximations have an easily definable area!

We will now treat this construction formally. Suppose that $f : L \to \mathbb{R}$ is a bounded function defined at all points on L. The fact that f is bounded means that $\sup_L f$ and $\inf_L f$ exist. As a consequence, we can take the $\sup$ and $\inf$ of f over any subset of L, and this will give us some number. Now suppose that $P \in \Delta_L$ is a partition of L with $P \equiv (a = x_0 < \ldots < x_n = b)$. We define the **upper** and **lower sums** of f with respect to P to be:

$$U(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1},x_i]} f$$
$$L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f.$$

If we are to claim that some value K represents the area under the curve of f on L, then we must certainly have:

$$L(f, P) \leqslant K \leqslant U(f, P).$$

In the case that f is everywhere non-negative, this apparent assertion is supported by the fact that the region defined by the upper rectangles is a superset of the shaded region, while the region defined by the lower rectangles is a subset of the shaded region. (We are using the *monotonicity* of area.) In the general case, compelling justification may be readily provided when several sentences are permitted. Hence, it follows that for every $P \in \Delta_L$, the interval $[L(f, P), U(f, P)]$ may be thought of as a range bounding the candidates that we would consider for an area.

One fundamental fact about these intervals is that refining P makes this window of values no wider.

Theorem *Suppose that* $P, Q \in \Delta_L$ *with* $P \leqslant Q$. *Then we have:*

$$L(f, P) \leqslant L(f, Q) \leqslant U(f, Q) \leqslant U(f, P).$$

Proof First off, we should make explicit something that was assumed in the discussion above: For any $P \in \Delta_L$, the fact that $L(f, P) \leqslant U(f, P)$ follows straight from the definition of upper and lower sums, since we always have $\inf_X f \leqslant \sup_X f$.

We will show that $U(f, Q) \leqslant U(f, P)$; the proof that $L(f, P) \leqslant L(f, Q)$ is similar. In fact, we will prove the result when Q is obtained from P by adding one additional point. This suffices since repeated applications of the argument will let us construct any $Q \geqslant P$, since the sets P and Q are finite. (Formally, we would justify this argument using induction.)

Thus, suppose that $P \equiv (a = x_0 < \ldots < x_n = b)$ and that

$$Q \equiv (a = x_0 < \ldots < x_k < x' < x_{k+1} < \ldots < x_n = b).$$

The difference between $U(f, P)$ and $U(f, Q)$ is that $U(f, P)$ has the sumand:

$$(x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} f,$$

while $U(f, P)$ has the sumands:

$$(x' - x_k) \sup_{[x_k, x']} f + (x_{k+1} - x') \sup_{[x', x_{k+1}]} f.$$

We have $\sup_{[x_k, x']} f \leqslant \sup_{[x_k, x_{k+1}]} f$ and $\sup_{[x', x_{k+1}]} f \leqslant \sup_{[x_k, x_{k+1}]} f$ by monotonicity. Thus the sum of the latter sumands are less than or equal to:

$$(x' - x_k) \sup_{[x_k, x_{k+1}]} f + (x_{k+1} - x') \sup_{[x_k, x_{k+1}]} f,$$

but reducing this expression yields the first sumand. □

Thus, the situation on out hands is that we have a directed set Δ_L, and for each $P \in \Delta_L$ we have an interval of possible values for the area under the curve. Moreover, these intervals get (weakly) narrower as we refine the partition. We will say that the area under the curve exists when these intervals hone into a unique number, and, in this case, we will call that number the **integral** of f on L.

Definition Let $f : L \to \mathbb{R}$ is a bounded function, and let $K \in \mathbb{R}$. We say that $\int_a^b f(x)\,dx = K$ provided that for every $A \in \mathcal{O}(K)$, there exists some $P \in \Delta_L$ with $[L(f,P), U(f,P)] \subseteq A$. If there exists some K such that $\int_a^b f = K$, then we say that f is **integrable** on L. ■

Note that this bears a close resemblence to the definition of $\lim_{x\to a} f(x)$. In that case, we considered the shrinking sets $f[B]$ for $B \in \mathcal{O}(a)$.

At the same time, there exists a different but equivalent formulation of this definition that is more amenable to calculation and the use of *limit laws*. This is the definition that we covered in class.

THE DEFINITION THAT WE'LL BE USING

Definition (Alternative) Let $f : L \to \mathbb{R}$ is a bounded function, and let $K \in \mathbb{R}$. We say that $\int_a^b f(x)\,dx = K$ provided that there exists a sequence of partitions P_n such that

$$\lim_n L(f, P_n) = \lim_n U(f, P_n) = K.$$ ■

Example Let $f(x) = x^2$. Show that $\int_0^1 f(x)\,dx$ exists and find its value.

Remark For reference, here are several useful formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Additionally, in order to save us the trouble of defining the same family of partitions in each example that we consider, we will define once and for all a particularly nice family of partitions. Fix $L = [a, b]$. For each $n \in \mathbb{N}$ with $n > 1$, we define the n-th uniform partition of L, denoted P_n, to be the partition with $n+1$ uniformly spaced points. More precisely, let $x_i = a + \frac{i}{n}(b-a)$ for $0 \leqslant i \leqslant n$.

Solution We begin by computing the upper and lower sums of f with respect to the uniform partitions P_n in order to get an idea of what the integral should be. Let $n \in \mathbb{N}$, with $n > 1$. We know that f is monotonically increasing on $[0, 1]$, thus the upper and lower sums become:

$$\begin{aligned}
U(f, P_n) &= \left(\frac{1}{n}\right)^2 \left(\frac{1}{n} - 0\right) + \ldots + \left(\frac{n}{n}\right)^2 \left(1 - \frac{n-1}{n}\right) \\
&= \frac{1}{n^3} + \frac{2^2}{n^3} + \ldots + \frac{(n-1)^2}{n^3} + \frac{n^2}{n^3} \\
&= \frac{1}{n^3}\left(1 + 2^2 + \ldots + (n-1)^2 + n^2\right) \\
&= \frac{n(n+1)(2n+1)}{6n^3} \\
L(f, P_n) &= \left(\frac{0}{n}\right)^2 \left(\frac{1}{n} - 0\right) + \ldots + \left(\frac{n-1}{n}\right)^2 \left(1 - \frac{n-1}{n}\right) \\
&= \frac{(n-1)n(2n-1)}{6n^3}
\end{aligned}$$

We can divide through to obtain:

$$\begin{aligned}
U(f, P_n) &= \frac{(1 + 1/n)(2 + 1/n)}{6} \\
L(f, P_n) &= \frac{(1 - 1/n)(2 - 1/n)}{6}
\end{aligned}$$

The limits $\lim_n L(f, P_n)$ and $\lim_n U(f, P_n)$ may now be calculated using limit laws. All that we need to know is that $\lim_n \frac{1}{n} = 0$ and the rest will follow via several applications of the laws.

$$\begin{aligned}
\lim_n L(f, P_n) &= \lim_n \frac{(1 - 1/n)(2 - 1/n)}{6} \\
&= \frac{1}{6} \lim_n (1 - 1/n) \lim_n (2 - 1/n) \\
&= \frac{1}{6} \left(1 - \lim_n 1/n\right) \left(2 - \lim_n 1/n\right) \\
&= \frac{1}{6}(1 - 0)(2 - 0) = \frac{1}{3} \\
\lim_n U(f, P_n) &= \lim_n \frac{(1 + 1/n)(2 + 1/n)}{6} \\
&= \frac{1}{6} \lim_n (1 + 1/n) \lim_n (2 + 1/n) \\
&= \frac{1}{6} \left(1 + \lim_n 1/n\right) \left(2 + \lim_n 1/n\right) \\
&= \frac{1}{6}(1 + 0)(2 + 0) = \frac{1}{3}.
\end{aligned}$$

Hence $\lim_n L(f, P_n) = \lim_n U(f, P_n) = \frac{1}{3}$, so $\int_0^1 f(x)\, dx = \frac{1}{3}$. $\square$