Top Grpd

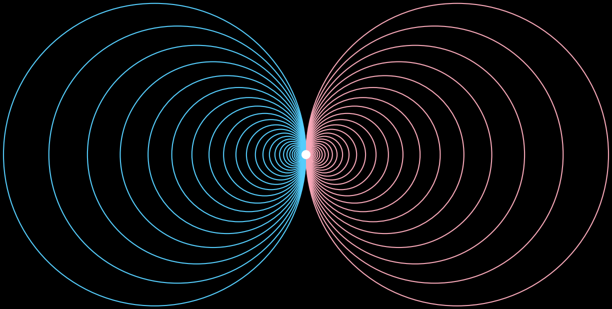
Mathematics Department Oral Exam

Minor Topic

$$\pi_1 : \mathsf{Top.} \longrightarrow \mathsf{Grp}$$

the fundemental group

	Top.	Grp
product	$X \times Y$	$G \times H$
coproduct	$X \vee Y$	$G * H$



$$\pi_1(X \vee Y) \ncong \pi_1(X) * \pi_1(Y)$$

We must ask that X and Y are *well-pointed*.

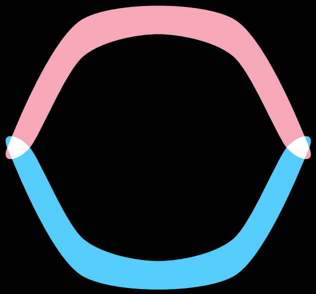
The Seifert–van Kempen Theorem:

Describes the fundamental group of a union of spaces.

In short: $\pi_1 : \mathsf{Top.} \to \mathsf{Grp}$ preserves *nice* colimits.

Big Caveat:

The intersections of the spaces must be path connected.



$$\pi : \mathsf{Top} \longrightarrow \mathsf{Grpd}$$

the fundemental groupoid

If $G \circlearrowright X$, we say X is a G-space.
$p : X \to X/G$ *(orbit space)*

In this case, πX is a G-groupoid.
$p : \pi X \to \pi X//G$ *(orbit groupoid)*

Under mild assumptions, $\pi(X/G) \cong \pi X//G$.

□ □ *End of intro.* □ □

Topic Outline:

1] Basic notions and category theory.

2] The statement and proof of the vKT.

3] Calculations: $\pi_1(S^1)$ and HNN-extensions.

4] The construction of colimits in Grpd.

1

Basic Notions

The Path Category:

Let X be a space. The path category PX has:

Objects: $\mathrm{x} : \mathrm{PX}$ the points of X.

Morphisms: $a : x \to y$ the maps $a : [0, r] \to X$, such that $a(0) = x$, $a(r) = y$.

Composition: The concatenation of paths. For $a : x \to y$, $b : y \to z$, we have $a \cdot b : x \to z$.

The zero length paths are units.
Concatenation is strictly associative.

Groupoid:

A *small category* in which every morphism is invertible.

We think of groupoids as algebraic objects, like groups.
We may perform set theoretic constructions on the object set.
This is a very *strict* notion.

A morphisms of groupoids $f : G \to H$ is a functor.
We then form Grpd, the category of groupoids.

The Fundamental Groupoid:

Formed as a quotient of the path category.

If $a, b : x \to y$ with $|a| = |b|$, then we say $a \sim b$,
if a is homotopic to b rel endpoints.

In general, for $a, b : x \to y$, $a \sim b$ means:
There exist constant paths r_y, s_y, such that $a \cdot r_y \sim b \cdot s_y$.

Taking path classes as morphisms, we form πX.

(There are several things to be checked.)

Homotopies of Functors:

Let $f, \ g : \mathcal{C} \to \mathcal{D}$ be functors.

A natural transformation $\alpha : f \Rightarrow g$ conists of:
for each $x : \mathcal{C}$, $\alpha_x : f(x) \to g(x)$, such that
for any $a : \mathcal{C}(x, y)$, $\alpha_x \cdot g(a) = f(a) \cdot \alpha_y$.

If each α_x is invertible, we get $\alpha^{-1} : g \Rightarrow f$.
We call such α a homotopy, and write $\alpha : f \simeq g$.

An equivalence of categories is a homotopy equivalence.

When f and g are morphisms of groupoids:
Any $\theta : f \Rightarrow g$ is a homotopy.

Homotopy Invariance of πX:

Let $X, Y : \mathsf{Top}$ and $f, g : X \to Y$.

THEOREM: If $f \simeq g$, then $\pi f \simeq \pi g$.

COROLLARY: If $X \simeq Y$, then $\pi X \simeq \pi Y$.

Deformation Retractions:

Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$.

Let $\theta : f \simeq g : \mathcal{C} \to \mathcal{E}$.
We say θ is $\mathrm{rel}\,\mathcal{D}$ if $\theta_x = 1$ for all $x : \mathcal{D}$.

We say $r : \mathcal{C} \to \mathcal{D}$ is a deformation retraction if
$ir \simeq 1_{\mathcal{C}} \,\mathrm{rel}\, \mathcal{D}$.

It follows that $ri = 1_{\mathcal{D}}$, so r is a homotopy equivalence.

Theorem: $\mathcal{D}$ is a deformation retract of $\mathcal{C}$
if and only if $\mathcal{D}$ is full and essentially wide.

(Non-constructive; this is equivalent to AC!)

Homotopy Types in Grpd:

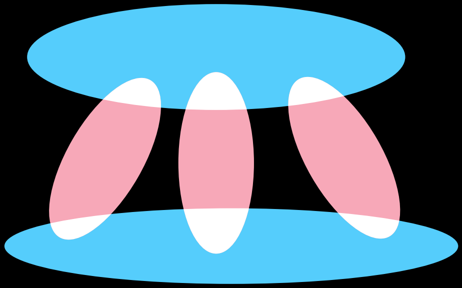
Let G be a connected groupoid (a fortiori non-empty).
Then G deformation retracts onto any $G(x)$.

Every connected groupoid has the homotopy type of a group.

Every groupoid has the homotopy type of a bundle of groups.

2

The van Kampen Theorem



A typical setting for vKT with $X \cup Y$.

Two Quick Definitions:

Let X be a space, and A be a set.

Interior cover of X:
A collection $\{U_\lambda\}_{\lambda\in\Lambda}$ of subspaces whose interiors cover X.

The pair (X, A) is connected:
A meets all path components of X.

We do not require A to be a subset of X.

Coequalisers:

Recall that given a parallel pair:

$$G \overset{f}{\underset{g}{\rightrightarrows}} H$$

A coequaliser of f and g is a morphism $p : H \to \mathrm{coeq}(f, g)$ such that $pf = pg$, and such that p is universal with respect to this property.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "p"] \arrow[dr, "q"'] & \mathrm{coeq}(f, g) \arrow[d, dashed, "q^*"] \\ & & X \end{tikzcd}$$

The van Kampen Theorem:

Let X be a space, $\{U_\lambda\}_{\lambda\in\Lambda}$ be an interior cover of X, and A be a set such that $(U_\lambda \cap U_\mu \cap U_\nu, A)$ is connected for all *(not necessarily distinct)* $\lambda, \mu, \nu \in \Lambda$.

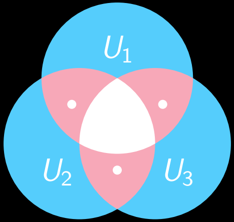
Then the following is a coequaliser in Grpd:

$$\coprod_{\lambda,\mu\in\Lambda} \pi\left(U_\lambda \cap U_\mu, A\right) \underset{\iota^2_{\lambda,\mu}}{\overset{\iota^1_{\lambda,\mu}}{\rightrightarrows}} \coprod_{\lambda\in\Lambda} \pi\left(U_\lambda, A\right) \xrightarrow{\iota_\lambda} \pi(X, A).$$

Why three-fold intersections?

Since $\pi(U_\lambda \cap U_\mu, A)$ appears in the diagram, having a condition on two-fold intersections seems necessary.

But this is not enough.



3
Calculations

The Interval Groupoid $\mathbb{I}$:

$$
\begin{tikzcd}
0 \arrow[loop left] \arrow[r, bend left, "\iota"] & 1 \arrow[loop right] \arrow[l, bend left, "\iota^{-1}"]
\end{tikzcd}
$$

An analogy:

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d, hook] & \{*\} \arrow[d] \\ {[0,1]} \arrow[r] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \{0,1\} \arrow[r] \arrow[d, hook] & \{*\} \arrow[d] \\ \mathbb{I} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

The Fundamental Group of S^1:

From van Kampen, we can immediately obtain:

$$\begin{tikzcd}
\{0,1\} \arrow[r, hook] \arrow[d, hook] & \mathbb{I} \arrow[d] \\
\mathbb{I} \arrow[r] & \pi(S^1, A)
\end{tikzcd}$$

We want to cut A down to one point:

$$\begin{tikzcd}
\mathbb{I} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
\pi(S^1, A) \arrow[r] & \pi_1(S^1, x_0)
\end{tikzcd}$$

HNN extensions:

Given a group G, two subgroups A, B,
and an isomorphism $\mu : A \cong B$, we want
a universal group homomorphism $\varphi : G \to G*_\mu$
such that $\varphi[A] \cong \varphi[B]$ via an inner automorphism.

Construction:

$$\begin{tikzcd} A \sqcup A \arrow[r, "{(i_A,\ i_B\mu)}"] \arrow[d, hook] & G \arrow[d, "\varphi"] \\ A \times \mathbb{I} \arrow[r, "\psi"'] & G *_\mu \end{tikzcd}$$

Let $t = \psi(e, \iota)$. Then for $a : A$,

$$\begin{aligned} t \cdot \varphi(a) \cdot t^{-1} &= \psi(e, \iota^{-1}) \cdot \psi(a, 1_0) \cdot \psi(e, \iota) \\ &= \psi(a, 1_1) = \varphi\mu(a) \end{aligned}$$

4

Construction of Colimits

Coproducts and Coequalisers:

Coproducts in Grpd are easy.

If we can show that coequalisers exist, then we have all colimits.

That is the goal of this section.

Two Steps:

Suppose that we have a parallel pair:

$$G \underset{g}{\overset{f}{\rightrightarrows}} H$$

1] We form a groupoid with the right object set.
(We perform a 0-identification on H.)

2] We form a quotient to get the right hom sets.
(We construct a quotient groupoid.)

Each of these constructions has an universal property.

0-identification morphisms:

Let $G : \mathsf{Grpd}$, $X : \mathsf{Set}$, and $\sigma : \mathrm{Ob}(G) \to X$.
For simplicity, we will here assume σ is surjective.

We want $U_\sigma(G) : \mathsf{Grpd}$ and a morphism $\overline{\sigma} : G \to U_\sigma(G)$
with $\mathrm{Ob}(\overline{\sigma}) = \sigma$ and satisfying the UP:
For any $f : G \to H$ such that $\mathrm{Ob}(f)$ factors through σ,
there is a unique $f^* : U_\sigma(G) \to H$ such that $f^*\overline{\sigma} = f$.

This is a direct analogue of a *quotient map* in Top.

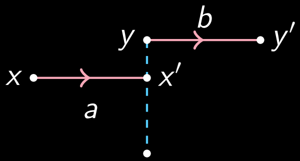
Reformulation of the UP:

Identify the sets $\mathrm{Ob}(G)$ and X with discrete groupoids.
Then $U_\sigma(G)$ is defined by a pushout diagram:

$$\begin{tikzcd}
\mathrm{Ob}(G) \arrow[r, "\sigma"] \arrow[d, "i"'] & X \arrow[d, "j"] \\
G \arrow[r, "\overline{\sigma}"'] & U_\sigma(G)
\end{tikzcd}$$

The Idea:

Morphisms that cannot be composed in G may become composable in $U_\sigma(G)$.



$$\sigma(x') = \sigma(y)$$

The product $\overline{\sigma}(a) \cdot \overline{\sigma}(b)$ should exist.

Reduced Word Construction:

The elements of $U_\sigma(G)$ are either:

identities $[]_x$, or

non-empty words $[a_1, \ldots, a_n]$ for $a_i : x_i \to x_i'$, where

1] $a_i \neq 1$

2] $x_{i-1}' \neq x_i$

3] $\sigma(x_{i-1}') = \sigma(x_i)$.

The product is given by concatenating and reducing.

(This is defined by induction on word length.)

$\overline{\sigma} : G \to U_\sigma(G)$ is $\overline{\sigma}(a) = [a]$ on non-identities.

Consequences of this Construction:

On a set A we may construct the free group $\lambda : A \to FA$.

On a graph Γ we may construct the free groupoid $\lambda : \Gamma \to \mathrm{Fr}\,\Gamma$.

Given two groupoids with overlaping object sets, we may form the free product $G * H$.
(vKT in the case of simply-connected intersection.)

Normal Subgroupoids:

A subgroupoid N of G is normal if:
N is wide, and
for any $a : x \to y$, $a \cdot N(y) = N(x) \cdot a$.

We will restrict our attention to totally-disconnected N.
If $\mathrm{Ob}(f)$ is injective, then $\ker(f)$ is totally-disconnected.

Quotient Groupoids:

We want to construct $p : G \to G/N$ such that $\ker(p) = N$.
G/N has objects $\mathrm{Ob}(G)$ and elements cosets $a \cdot N(y)$.

Let R be a collection of elements in the point groups of G.
Have $N(R)$ – the smallest normal subgroup containing R.

UP of $G/N(R)$:
Any $f : G \to H$ which annihilates R uniquely factors through $p : G \to G/N(R)$.

The Construction of Coequalisers:

First we define $\sigma : \mathrm{Ob}(H) \to X$ by by coequalising in Set:

$$\mathrm{Ob}(G) \underset{\mathrm{Ob}(g)}{\overset{\mathrm{Ob}(f)}{\rightrightarrows}} \mathrm{Ob}(H) \xrightarrow{\sigma} \mathsf{coeq}(\mathrm{Ob}\, f, \mathrm{Ob}\, g)$$

Now $\overline{\sigma} f(a)$ and $\overline{\sigma} g(a)$ lie in the same hom set. Define:

$$R(x) = \{\overline{\sigma} f(a) \cdot \overline{\sigma} g(a)^{-1} \mid a : G(y, y'),\ \sigma(y) = x\}$$

Finally, form:

$$p : U_\sigma(G) \to U_\sigma(G)/N(R).$$