Lie Theory as Illustrated by $SU(2)$

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1 Introduction

Similar to the history of how Galois groups were introduced to study polynomial equations $p(x) = 0$ and symmetries that act on the roots of polynomials, Lie groups were introduced to study differential equations. The idea is that sometimes, there are "symmetries" appearing which give additional solutions. For example, if $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation $f' - f = 0$, so does $f(x - c)$ for any $c \in \mathbb{R}$. Thus, this equation exhibits translational symmetry.

However, Lie theory has since developed into a subject in its own right, with deep connections to different parts of math and physics. In this note, we will briefly survey some important themes by studying the concrete example of the Lie group $SU(2)$.

2 $SU(2)$ and its Significance

$SU(2)$ can firstly be viewed as the set of 2×2 unitary matrices with determinant equal to 1; i.e. 2×2 complex matrices satisfying $A^*A = I$ and $\det A = 1$. One can check that if A, B are matrices satisfying these equations, then so does their product AB and their inverses A^{-1}, B^{-1}. Hence, under matrix multiplication, $SU(2)$ has the structure of a **group**.

An alternative but equivalent view is to study **quaternions**. These are a natural extension of the complex numbers but instead of introducing only one number i which satisfies $i^2 = -1$, we introduce three imaginary units: i, j, k which satisfy some simple relations: $i^2 = j^2 = k^2 = -1$ and $ij = k, jk = i, ki = j$. Thus, quaternions are modeled on the vector space $\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}\langle i \rangle \oplus \mathbb{R}\langle j \rangle \oplus \mathbb{R}\langle k \rangle$ but also have a multiplicative structure.

As a space, the unit length quaternions form the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$. So these unit quaternions not only form a group, they also have the structure of a **smooth manifold** which is compatible with the group structure. These are the conditions needed to make S^3 a **Lie group**. Intuitively, these properties mean that we have a space on which we can do calculus, such as defining differentiable and integrable objects on S^3. Moreover, the group multiplication is also a differentiable operation. It turns out that there is a Lie group **isomorphism** $S^3 \to SU(2)$; practically speaking, this is what formalizes the idea that S^3 and $SU(2)$ describe the same object.

Before focusing on the properties of $SU(2)$, it is worth stating that it has quite the role in the Standard Model of Particle Physics. In coarse terms, it was discovered that various elementary particles display a property known as **spin** and that $SU(2)$ governs the spin behavior of some of these particles. The formal way to study this is via the **representations** of $SU(2)$ which describe ways that $SU(2)$ acts linearly on vector spaces representing particle spin states. This action respects the group operation of $SU(2)$. Moreover, even beyond particle physics, $S^3 \cong SU(2)$ is used to compute the rotation of objects in space. For instance, a video game designer might want to change the camera angle which involves a rotation of the scenery. Or astronauts may analyze the rotation of, say, the ISS during docking procedures.

3 Properties of $SU(2)$

As a topological space, $SU(2)$ is compact, connected, and simply connected. Moreover, there is a relationship between $SU(2)$ and $SO(3)$ which is the group of orientation-preserving isometries on $\mathbb{R}^3$; these are rotations that fix the origin. A rotation of $\mathbb{R}^3$ is determined by an axis and an angle but there is some ambiguity between clockwise and counter-clockwise rotation. That is, one can rotate about the vector v by an angle θ but the same rotation is also described by $-v$ and $-\theta$.

$SU(2)$ clarifies this ambiguity. More formally, when we use the quaternionic view, we can view $\mathbb{R}^3 = \mathbb{R}\langle i\rangle \oplus \mathbb{R}\langle j\rangle \oplus \mathbb{R}\langle k\rangle$ as the purely imaginary quaternions. Then, taking $t \in S^3, q \in \mathbb{R}^3$, the conjugation map $q \mapsto t^{-1}qt$ is interpreted as a rotation because $t^{-1}qt$ lands back in the imaginary quaternions and its length is preserved. Hence, we have an isometry. Moreover, morally, this is an orientation-preserving isometry because S^3 being connected means we can find a path from 1 to t. 1 gives the identity isometry and is orientation-preserving. Because of continuity, all along the path, the corresponding isometries must also be orientation-preserving.

Note that $-t$ gives the same rotation but $t \neq -t$. Therefore, we obtain a 2-to-1 Lie group morphism $SU(2) \to SO(3)$ which says that locally, these Lie groups are equivalent. This map has many elegant topological, differentiable, and algebraic properties and also plays a role in the Standard Model. Once again, in coarse detail, the irreducible representations of compact Lie groups often arise in physics. In this case, the irreducible representations of $SO(3)$ are called n-spin representations while those of $SU(2)$ are called $\frac{n}{2}$-spin representations; n is related to the dimension of the associated vector space of particle states and $SU(2)$ has "twice" as many representations as $SO(3)$. The $SO(3)$ representations are enough to study the emission spectrum of the hydrogen atom, for instance, but not enough to study the quantum properties of, say, electrons. For that, one needs to upgrade to $SU(2)$.

This discussion is also related to some of the algebraic properties of $SU(2)$. Since matrix multiplication in $SU(2)$ is not commutative, $SU(2)$ is, by definition, **nonabelian**. One consequence of this is that its irreducible representations have dimension greater than one. This confirms that indeed, it makes sense to have $\frac{n}{2}$-spin representations where n can be any positive integer. Also, since the covering map sends $\pm t$ to the same rotation, the map has $\{\pm I\}$ as kernel which shows that $SU(2)$ is not a simple Lie group.

4 The Lie Algebra $\mathfrak{su}(2)$

An important concept introduced in calculus is that given a differentiable function $f : \mathbb{R} \to \mathbb{R}$, we can find the **tangent line** to the graph of f at any given point. More generally, for smooth manifolds, we can define a **tangent space** at each point on the manifold which all have many useful features. For instance, the tangent line represents a linear and "infinitesimal" approximation of the graph and the same can be said of the tangent space of a manifold. Concretely, the tangent space at a point is a vector space with the same dimension as the underlying manifold.

However, when studying the tangent space at the identity element of a Lie group, we have even more structure than that of a vector space. There is an operator called the **Lie bracket** that is inherited from the Lie group.

In the case of $SU(2)$, its Lie algebra $\mathfrak{su}(2)$ is a real 3-dim vector space which can be viewed as the vector space of 2×2 complex matrices that are skew-Hermitian and traceless. That is, $A^* = -A$ and the sum of the diagonal elements is zero. Here is a standard basis.

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

For general matrix Lie algebras such as this, it is important to remember that the product of two matrices might not be in the vector space. However, the commutator is: if A, B are in the Lie algebra, then $[A, B] := AB - BA$ is in the Lie algebra. This $[\cdot, \cdot]$ is the Lie bracket mentioned earlier and it gives extra structure which enriches the object. For example, one can now study Lie algebra representations which describes how a Lie algebra acts linearly on a vector space; the action respects the Lie bracket.

There is quite an important relationship between a Lie group G and its Lie algebra $\mathfrak{g}$. In general, there is an exponential map $\exp : \mathfrak{g} \to G$ and in the case that G is compact and connected, this map is surjective. When G is additionally simply connected and **semisimple**, there is a 1-1 correspondence between the representations of the Lie algebra and the Lie group. This is extremely useful since a Lie algebra is a vector space and hence, a study of its representations tends to be more tractable than a study of the representations of the Lie group. Put another way, it is amazing that something linear which serves as an infinitesimal approximation can capture so much about the underlying curved Lie group. This is illustrated in the case of $SU(2)$ and $\mathfrak{su}(2)$ where the exponential map is surjective and gives a way of relating their representations. Moreover, it is easy to define the exponential of matrices. For example, when $t \in \mathbb{R}$,

$$\exp(tX) = I + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + ... = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in SU(2).$$

As a final demonstration, we can even use Lie algebras to say something about the topology of the underlying Lie groups. Since, the Lie algebra is a local approximations of the Lie group, the covering map $SU(2) \to SO(3)$ from earlier tells us that $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ are isomorphic. On the other hand, since the representations of $SU(2)$ and $SO(3)$ are not in 1-1 correspondence and we know $SU(2)$ is simply connected, we conclude that $SO(3)$ is not simply connected.

5 Concluding Remarks

There is much more that can be said about Lie theory but hopefully, this note demonstrates that it is an area of math that sees the fruitful interplay of many subjects: calculus, linear algebra, group theory, representation theory, and topology, to name a few. The general theory is very rich but even understanding one example, such as $SU(2)$ and its Lie algebra $\mathfrak{su}(2)$, provides much insight.

Indeed, one can actually classify all semisimple Lie algebras over fields of characteristic zero by studying their finite dimensional complex representations. The traditional starting point for this is to study the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ which arises as a basic building block or at least, as an illustration of general patterns for all the semisimple Lie algebras. And it turns out, $\mathfrak{sl}(2, \mathbb{C})$ is Lie algebra isomorphic to the complexification of $\mathfrak{su}(2)$; i.e. $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{su}(2) \otimes \mathbb{C}$.