Inductive White and the $(N-1)$ Dwarves

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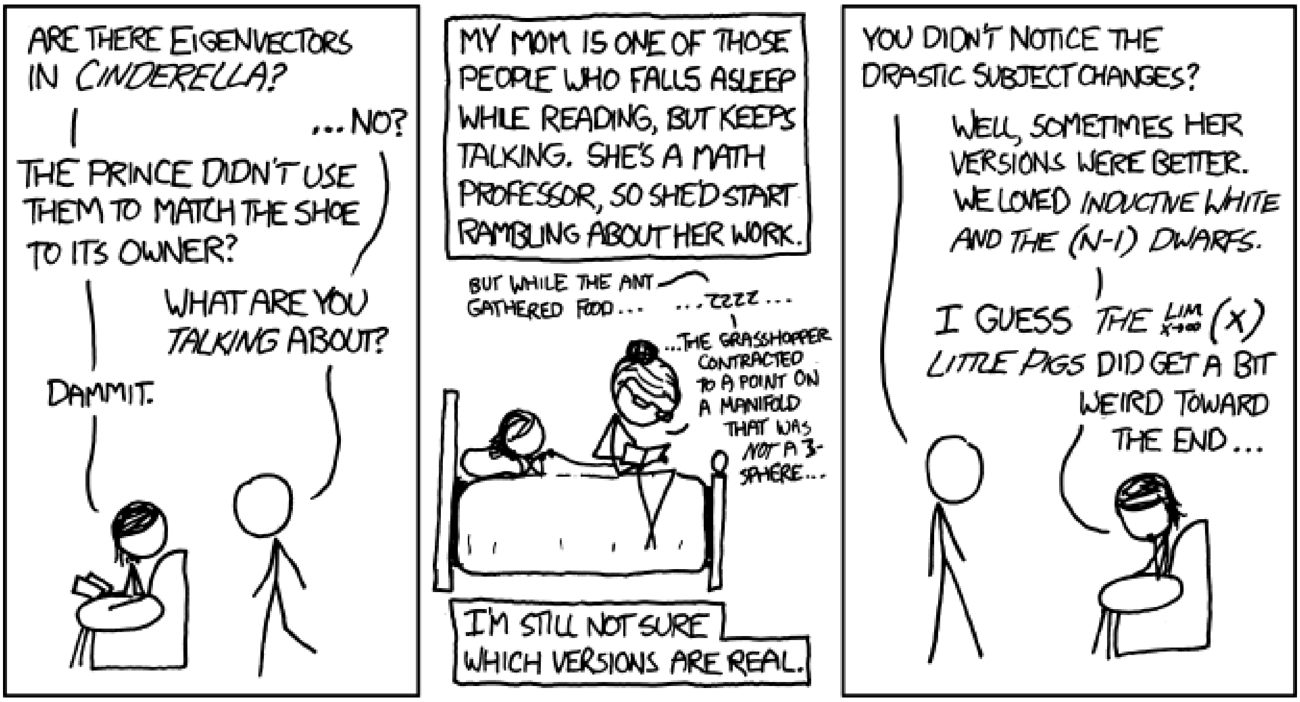
Randall Munroe, the creator of the popular webcomic xkcd has this comic about fairy tales and math mixing together. This is quite an old webcomic, number 872. However, when I looked to see if anyone has actually written a version of the fairy tale, I didn't find any. Of course, I didn't look that hard; I was already thinking of my own version of the story.



Thus, I decided to try writing a few of my own spins on classic fairy tales. I tried to stick somewhat close to the original stories. This is surely contestable because in the case of Cinderella, say, there are many versions, including the Brothers Grimm and also Disney. I would lean towards the Disney version both because it's more well-known and also, less gruesome. I also wanted to stick to Munroe's original idea. I'm not sure how to incorporate eigenvectors into Cinderella but there you go.

While reviewing Disney's version of Snow White, I realized there were a lot of details I did not recall. First, I knew Snow White was Walt Disney's earliest film but I didn't expect it to be from 1937. Secondly, I didn't remember the wicked queen being Snow White's own mother nor do I remember her asking for Snow White's heart in a box. Lastly, I didn't realize that it was a whole year after her sleep that the prince arrived.

1 The Fairy Tale

Once upon a time, in a far away land, there the Well-Ordering Queen. She had a daughter named Inductive White because her fair skin was as white as inductive. The queen, however, was quite a vain queen and feared tat Inductive White would become more fair than her. So she had Inductive White do menial tasks like showing, via induction, that

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}.$$

This is menial because you can show this much more quickly without induction. The Well-Ordering Queen would also ask her magic mirror daily

“Mirror, mirror, on the wall
Who's the fairest of them all?

It had always been that the mirror responded with, “You, Well-Ordering Queen, are the fairest of them all.” But one day, the mirror responded with: “Inductive White is the fairest of them all.”

The Well-Ordering Queen was furious. “How could Inductive White be the fairest of them all? The well-ordering principle is equivalent to the principle of mathematical induction. We should, at most, be equivalent in fairness! All she ever does is go around muttering about how a sequence of propositions $\{P(n)\}$ such that $P(n) \implies P(n+1)$ is like a line of dominoes, all ready to fall by a base case domino.”

Had the mirror offered an opinion, perhaps it would have said that induction is directly used far more often than the well-ordering principle.

So the Well-Ordering Queen decided to send Inductive White into the forest to be killed by the huntsman, Peano. “Bring me her heart—that is, her base case— in a jeweled box!” declared the queen. But when it came down to it, Peano couldn't bring himself to kill Inductive White because it would leave a problematic gap in the axioms. Tearfully, Peano begs for her forgiveness, revealing the queen's wicked plan and urges her to flee into the Modal Logic Woods.

Due to the fact that she is of 2nd-Order logic, the Modal Logic Woods was a terrifying place for her. However, the princess was soon met by some woodland creatures, such as $(\Diamond p \to \Box\Diamond p)$ (“If p is possible, then p is necessarily possible.”). They led her to a cottage deep in the woods.

The door to the cottage was unlocked because who would dare wander around in the Modal Logic Woods? So Inductive White ventured inside and found seven small chairs in the dining room. “What an untidy place this is!” Inductive White declared. She began to clean the cottage and after finishing, she fell asleep in one of the upstair beds.

Unbeknownst to Inductive White, the cottage is the home of $(N-1)$ dwarfs who spend their day in a nearby mine, searching for precious well-formed formulas. For example, $(P \implies P)$ is not a precious well-formed formula.

When the $(N-1)$ dwarfs returned home, they found Inductive White upstairs, asleep across three of their beds. When Inductive White woke up, she found all $(N-1)$ dwarfs at the bedside so introduced herself. Some of the dwarfs immediately liked Inductive White but others, such as Carnap and Ayer, were not so keen.

But after Inductive White gave them a logic problem, they all warmed up to her. The problem is as follows.

“Suppose that all of you dwarfs are wearing green dwarf hats and you can of course, see every other dwarf's hat but not your own. For some reason, you don't talk about hats or the color green because, well, every since Noam Chomsky announced that ”Colorless green ideas sleep furiously,” you have shunned using the word “green” in everyday language. Also, there are certainly no mirrors around here because the Well-Ordered Queen is the only one with a mirror. However, for some peculiar reason, if any one of you realize that you are wearing a green hat, you will eat your own hat for supper that night. One day, I arrive and I learn about the situation. I say something that I think does not give you any new information: “At least one of you is wearing a green hat.” What will happen?”

The dwarfs mused on that one for awhile but eventually came to the right conclusion. On the $(N-1)$ day, they all eat their hat. Meanwhile, the Well-Ordered Queen discovered that Inductive White was still alive when the mirror again answers that Inductive White was the fairest in the land.

Thus, the queen decided to enter the Modal Logic Woods. Using a potion, she disguised herself as the innocuous Axiom of Extensionality in Zermelo-Fraenkel set theory. The queen also brought with her a very dangerous apple, Russell's Apple, which throws any person who eats it into such an infinite loop of confusion that they become unable to do anything else. The queen learned that the loop could be broken by something sufficiently powerful. Not true love's kiss or something silly like that but say, an impressive mathematical result. Not knowing that Inductive White had in fact met a prince named Prince Zorn, she felt her plan was fail-proof.

Thus, the Well-Ordered Queen arrived at the cottage while the dwarfs were out mining. The creatures of the Modal Logic Woods recognize the queen through her Axiom of Extensionality disguise and tried to chase her away. But through some trickery, the queen got Inductive White to bring her inside. The queen then fooled Inductive White into eating Russell's apple. "Eating this apple will give you the power to determine which sentences are decidable!" lied the queen.

The dwarfs returned with the woodland creatures as the queen left but were too late. Inductive White was already muttering to herself, unaware of all that was around her. "Well, the barber cuts the hair of someone if and only if that person does not cut their own hair. So the barber should not cut his own hair. But then, that means he qualifies as one who should have his hair cut by himself. So then the barber does cut his own hair. But then this disqualifies him..."

The dwarfs and woodland creatures gave chase, with $(\Box P \rightarrow \Box\Box P)$ leading the charge. Eventually, the queen was trapped on a cliff. She tried to eliminate her pursuers, ironically, through an inductive procedure: use the well-ordering theorem, eliminate the smallest in the set, and then repeat. But there was no obvious ordering, or even partial ordering. Thus began an ineffective skirmish between the queen and the dwarfs and modal creatures. However, eventually, the queen made the fatal error of mistaking $\Box(P \rightarrow \Diamond P)$ and $(P \rightarrow \Box\Diamond P)$ as the same creatures. While distracted by $\Box(P \rightarrow \Diamond P)$, $(P \rightarrow \Box\Diamond P)$ pushed the Well-Ordering Queen off the cliff and to her death. This also demonstrated that physical attacks fare better than logical ones.

Following this, the dwarfs returned to their cottage and found Inductive White seemingly unconscious. Indeed, her body had ceased to do all but the barest minimum to keep her alive. All her energy was devoted to resolving Russell's Paradox. The dwarfs were unwilling to bury her out of sight in the ground so instead, they did the weird thing of placing her in a glass coffin in a clearing in the forest as a display, similar to what happened to the bodies of Joseph Stalin and Mao Zedong.

Together with the woodland creatures, the dwarfs kept watch over Inductive White. A full year later, Prince Zorn was back in the Modal Logic Woods and found Inductive White in the coffin. Saddened by her apparent death, Prince Zorn decided to pay his respects by proving, in her honor, that every field has an algebraic closure and that in a ring with identity, every proper ideal is contained in a maximal ideal. As he turned to leave, the coffin opened and Inductive White sat up. Apparently, this was sufficient for breaking the power of Russell's Apple.

"I've got it!" exclaimed Inductive White. "We must develop type theory to address this paradox." She then noticed Prince Zorn. "Oh hello, Prince Choice. I mean, Prince Hausdorff. I mean, Prince Zorn."

The dwarfs and the creatures then accompanied Inductive White and Prince Zorn out of Modal Logic Woods and into the Well-Ordering Castle. "Oh! Dear prince, you appear to be Prince Well-Ordering!" said Inductive White. "Not to worry, the Well-Ordering Principle and the Well-Ordering Theorem are different statements," assured the prince. And they lived happily ever after, trying to develop type theory until they had a son, named Kurt, who proved their endeavor was fruitless. The end.

2 Comments

- Students of modal logic often confuse $\Box(P \to \Diamond P)$ with $(P \to \Box\Diamond P)$.
- The problem with the green hats is a classic problem that usually is formalized with green-eyed dragons that turn into sparrows if they realize they have green eyes. The solution is pretty easy if one considers the case where there is only $n = 1$ dragons, then $n = 2$, then $n = 3$. Hopefully, it convinces you that on the Nth day, the dragons all simultaneously turn into sparrows.
- The Axiom of Extensionality simply states that two sets are the same if and only if they have exactly the same elements.
- The queen's lie: There is no algorithm for determining when something is decidable. Indeed, there are many undecidable situations.
- Of course, Zorn's Lemma is equivalent to the Axiom of Choice which is equivalent to Haussdorff's Maximal Principle which is equivalent to the Well-Ordering Theorem.
- Bertrand Russell and Alfred North Whitehead attempted to develop type theory in order to form the foundations for mathematics. They produced, in my opinion, some horrible three volumes of the *Principia Mathematica* which, as Kurt Gödel showed later, is still incomplete (in the logic sense).