Compactness of the Moduli Space $\mathcal{M}$ of Finite Energy Floer Solutions (when $\pi_2(\omega) = 0$)

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1 Morse Theory Model

Let V be a smooth manifold with a Morse function f and some Riemannian metric. We may consider the space of all smooth maps $u : \mathbb{R} \to V$ which satisfy $du/ds = -\nabla f$ and have finite energy: $\int_{\mathbb{R}} |du/ds|^2 < \infty$. Let this space be called $\mathcal{M}$. It's not hard to show that this definition of energy coincides with another one we've seen:

$$E(u) = -\int_{\mathbb{R}} u^* df.$$

We have that

$$u^* df = d(f(u(s))) = df(\nabla_u f(s))\, ds = \langle \nabla_u f(s), \nabla_u f(s) \rangle\, ds = \left\| \frac{du}{ds} \right\| ds.$$

With finite energy and the Fundamental Theorem of Calculus, this means that $\lim_{s \to -\infty} f(u(s)) - \lim_{s \to +\infty} f(u(s)) < \infty$. Thus, each of the limits needs to be finite and this means then that $\lim_{s \to -\infty} u(s) = a$ and $\lim_{s \to +\infty} u(s) = b$ for some critical points a and b of f. Thus, what we've shown is that

$$\mathcal{M} = \bigcup_{a, b \in \mathrm{Crit}(f)} \mathcal{M}(a, b).$$

When V is compact, then **all** the maps have finite energy. It's an Arzelà-Ascoli argument which establishes the compactness of this space; we can obtain a uniform bound on the derivatives since a solution u has $du/ds = -\nabla f$; the bound is just $\|\nabla f\|$.

But one wonders: "Let's construct a sequence of maps u_k that represent trajectories, say, between critical points x and z that have index differing by 2. Let's make it so that the image of these u_k's, that is, the flow lines, appear to be converging towards a broken trajectory that connects x to y by a map v, and then y to z via a map w. $\mathcal{M}$ does **not** have such broken trajectories in it; how can $\mathcal{M}$ be compact?"

There are two things to say here:

1. It is dangerous to look at the images of the u_k and conclude that their limit has to have the limit of the image. It is not true in general. We need to look at the convergence of the **maps**. We see this with bubbling examples: looking at the image doesn't lead to the correct conclusion about the maps.

2. The broken trajectory which is made of v and w is made of five parts: the point x, the map v, the point y, the map w, and the point z. The converging subsequence of u_k will have to converge to precisely one of these. One might protest: "But say they're going to

v. There are points in the image of these u_k which are far from the image of v! How do they end up on the image of v?" The answer is that the image of v is an open thing so it's homeomorphic to an infinitely long line. Those points that are far away, will get pulled onto the image of v eventually in the limit. This is one of the oddities of convergence in C^∞_{loc}. A simple analogous picture to keep in mind is that of smooth functions β_k which is 0 on $(-\infty, k)$ and 1 on $(k+1, +\infty)$. Then, as $k \to +\infty$, it either converges to the constant function 0 but as $k \to -\infty$, it converges to the constant function 1. Thus, depending on the direction in which we translate, the convergence changes. This is similar for the case of Morse trajectories and also Floer trajectories.

So why do we add broken trajectories when we take the quotient $\mathcal{L} = \mathcal{M}/\mathbb{R}$? When taking the quotient, we want to identify all translations of these maps $u : \mathbb{R} \to V$. After all, translating u by s gives a **different** element in $\mathcal{M}$. We want to identify these. Thus, taking the quotient means that we need to somehow keep track of all the information about translations. In particular, it means if a sequence u_k admits subsequences that can be translated towards different limits, we group those limits into one object in $\mathcal{L}$. In our example above, that means x, v, y, w, or z.

To reiterate, what does it means for $u_k \in \mathcal{L}$ to converge to a broken trajectory? **It means exactly that for each piece of the broken trajectory, there exists a subsequence, still call it u_k, and some sequence s_k, such that $u_k \cdot s_k$ converges to that piece of the broken trajectory.**

This is an unintuitive concept that we're dealing with: quotienting by $\mathbb{R}$ means we have to be **more** careful in keeping track of **more** information than pre-quotienting. The sliding bump functions β_k that we saw earlier represent this. We can fix any $r \in \mathbb{R}$ and choose a sequence s_k to translate by so that $\beta_k \cdot s_k$ converges to a smooth function

$$\beta(s) = \begin{cases} 0 & s < r \\ \text{something in } [0,1] & s \in [r, r+1] \\ 1 & s > r+1 \end{cases}$$

2 Background for the Floer Theory Case

Let (M, ω) be a compact symplectic manifold and $H : M \times \mathbb{R} \to \mathbb{R}$ a time dependent Hamiltonian. Let $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ be the associated action functional on the loop space $\mathcal{L}M$. Recall that the critical points (which are loops) of $\mathcal{A}_H$ are precisely the solutions of a time-dependent Hamiltonian system. In this note, I try to explain why the moduli space $\mathcal{M}$ of contractible C^∞ solutions of the Floer equation with finite energy is compact. Recall the Floer equation:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H_t(u) = 0 \tag{2.1}$$

A solution $u : \mathbb{R} \times S^1 \to M$ is precisely a trajectory of $-\nabla \mathcal{A}_H$, the negative gradient of the action functional. The reason we're interested in the trajectories of the gradient is because we wish to find a Morse theoretic solution to Arnold's conjecture. As such, we need to construct a chain complex with critical points generating the modules and trajectories defining the differential. The differential should be defined by counting trajectories but in order to do so, we need the moduli space of trajectories to be compact.

We also need to define the notion of energy for a solution. From Morse theory, if $u : \mathbb{R} \to M$ is a solution to the differential equation $\partial u/\partial s + X(u) = 0$, then we defined above the energy as $E(u) = -\int_{-\infty}^{+\infty} u^* df$. Then, if u limits to critical points a and b, the energy is simply $f(a) - f(b)$

by the Fundamental Theorem of Calculus. Thus, we try to achieve the same thing here. Define energy by

$$E(u) = -\int_{-\infty}^{+\infty} \frac{d}{ds}\mathcal{A}_H ds = \int_{\mathbb{R}\times S^1} \left|\frac{\partial u}{\partial s}\right|^2 ds\, dt = \int_{\mathbb{R}\times S^1} \omega\left(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}\right) ds\, dt.$$

The Floer equation gives us that $J \cdot \partial u/\partial s = \partial u/\partial t + X_t(u)$. The energy is positive and equals zero only if u is independent of s; i.e. it is just a loop which is a critical point of $\mathcal{A}_H$. Then we have that if u has limits x, y which are critical points of the action functional, then $E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y)$.

Major Claim: If u is a trajectory with finite energy, then there exists x and y which are critical points of $\mathcal{A}_H$, such that $\lim_{s\to-\infty} u(s,t) = x(t)$ and $\lim_{s\to+\infty} u(s,t) = y(t)$. This takes some showing but morally, if u has finite energy, it requires that $\partial u/\partial s \to 0$. Then, in the limit, the Floer equation "limits" to the Hamiltonian equation:

$$J(u)\frac{\partial u}{\partial t} = -\nabla_u H_t \Longleftrightarrow \frac{\partial u}{\partial t} = X_t(u).$$

Because there are only finitely many critical points for $\mathcal{A}_H$, this means that there exists a $C > 0$ such that every trajectory u of finite energy satisfies $-C \leq \mathcal{A}_H(u_s) \leq C$; i.e. the action functional is **uniformly bounded** along every u by the same constant C. Moreover, $0 \leq E(u) \leq C$ because of this bound. So there is also a **uniform bound** on the energy. This is **not** enough to show the moduli space $\mathcal{M}$ is compact but is useful in one of the steps.

We also need the following assumption on our symplectic manifold: $\pi_2(\omega) = 0$, meaning that for all smooth $f : S^2 \to M$, $\int_{S^2} f^*\omega = 0$.

3 Idea of Proof

Let us periodically extend everything from $\mathbb{R}\times S^1$ to $\mathbb{C} \cong \mathbb{R}^2$. Suppose that $\mathcal{M}$ is not compact. This means, in particular, that there is no uniform bound on the gradients of the trajectories. For, if there were, then Arzelà-Ascoli would give us C^0 (sequential) compactness and by some elliptic regularity results, any sequence would admit a subsequence that not only converges to a C^∞ limit but does so in the C^∞_{loc} topology, thereby granting us compactness.

Thus, we assume we have a sequence u_k and a sequence (s_k, t_k) such that $\|\nabla_{(s_k,t_k)} u_k\| \to \infty$. We will, from these sequences, construct another sequence, defined on enlarging disks, whose limit contradicts $\pi_2(\omega) = 0$.

The idea is to carefully rescale and translate the u_k. First, define $\hat{u}_k(s,t) = u_k(s+s_k, t+t_k)$. Then, $\hat{u}_k(0,0) = u_k(s_k, t_k)$. Next, choose R_k such that $R_k \to \infty$ and $\epsilon_k \to 0$ (all positive) such that $\epsilon_k R_k \to \infty$. So the ϵ_k get smaller but not fast enough to stop the R_k from diverging to $+\infty$. We define $v_k(s,t) = \hat{u}_k(s/R_k, t/R_k)$, $\|\nabla v_k\| = 1$. So we recentered and rescaled the u_k.

We use the **half max lemma** to produce a sequence (ϵ_k) with $0 < \epsilon_k \to 0$ such that $\epsilon_k R_k \to \infty$ and $B(0, \epsilon_k R_k)$, $\|\nabla_{(s,t)} v_k\| \leq 2$. So we have a uniform bound on these v_k so by Arzela-Ascoli, they converge to some $v : \mathbb{C} \to M$. By elliptic regularity, $v \in C^\infty_{loc}$.

v has finite energy, has finite and nonzero symplectic area, and most importantly, is holomorphic. To see holomorphicity, note that the v_k satisfy:

$$\frac{\partial v_k}{\partial s} + J_{t+t_k}(v_k)\frac{\partial v_k}{\partial t} = -\frac{\nabla H_{t+t_k} v_k}{R_k}.$$

The translations don't really affect the equation but the R_k diverge to infinity so the right hand side vanishes in the limit and we're left with the Cauchy-Riemann equation. Since v is holomorphic, we may use a removable singularity result to extend v from $\mathbb{C}$ to S^2. But then we have a contradiction against the assumption of $\pi_2(\omega) = 0$:

$$\int_{S^2} v^*\omega \neq 0.$$

Remark 1: This is the formation of a bubble. We can picture a sequence of **expanding** 2-disks glued onto M by v. It turns out the boundary of the disks **shrink** to a point. So the limiting case (in C^∞_{loc}) is an S^2 "stuck" onto M.

Remark 2: The Half Max Lemma and this general proof is used often to show a moduli space of solutions is compact under some assumptions. See Chen, Donaldson, or Uhlenbeck's work.

4 More Detailed Proof

The compactness of $\mathcal{M}$ follows from the following proposition:

Proposition 4.1 (6.6.2)**.** *Under the assumption that $\pi_2(\omega) = 0$, there exists a constant $A > 0$ such that for all $u \in \mathcal{M}$ and all $(s,t) \in \mathbb{R} \times S^1$, $\|\nabla_{(s,t)} u\| \leq A$.*

This proposition gives us a uniform bound on the derivatives of the u since there is a bound on the gradients. Thus, by Arzelà-Ascoli, we have that the closure of $\mathcal{M}$ is compact in the space of all continuous maps. Thus, any sequence u_k admits a subsequence that converges to a u_0 in $C^0_{\text{loc}}(\mathbb{R} \times S^1, M)$. The next step is to show that in fact, u_0 is C^∞ and a solution of the Floer equation. Lastly, we want to ensure that the u_k also converge to u_0 in the C^∞ topology. This is proven by the fact that C^0_{loc} and C^∞_{loc} coincide on $\mathcal{M}$. So $\mathcal{M}$ is compact in the C^∞ topology.

Let's sketch a proof of Proposition 6.6.2. It is convenient to think of a solution $u : \mathbb{R} \times S^1 \to M$ as a periodic function in t, $u : \mathbb{C} \cong \mathbb{R}^2 \to M$.

Suppose that Prop. 6.6.2 is false. So there exists a sequence u_k in $\mathcal{M}$ and a sequence $(s_k, t_k) \in \mathbb{R}^2$ such that $\lim_{k\to\infty} \|\nabla_{(s_k,t_k)} u_k\| = +\infty$ (call these terms R_k). Let (ϵ_k) be a sequence of positive numbers tending to 0 such that $\lim_{k\to\infty} \epsilon_k R_k = +\infty$. We apply the following lemma to the function $g = \|\nabla u\|$.

Lemma 4.2 (6.6.3: Half Maximum Lemma)**.** *Let $g : X \to \mathbb{R}^+$ be a continuous function on a complete metric space. Let $x_0 \in X$ and let $\epsilon_0 > 0$. There exists a $y \in X$ and an $\epsilon \in (0, \epsilon_0]$ such that*

1. *$d(y, x_0) \leq 2\epsilon$.*
2. *$\epsilon_0 g(x_0) \leq \epsilon g(y)$*
3. *$g(x) \leq 2g(y)$ for all $x \in B(y, \epsilon)$.*

Proof. If $g(x) \leq 2g(x_0)$ on the entire ball $B(x_0, \epsilon_0)$, then we just need to let $y = x_0, \epsilon = \epsilon_0$. If this is not the case, then there is an $x_1 \in B(x_0, \epsilon_0)$ such that $g(x_1) > 2g(x_0)$. Let $\epsilon_1 = \epsilon_0/2$. We then have $\epsilon_1 g(x_1) \geq \epsilon_0 g(x_0)$. If x_1 and ϵ_1 are suitable, we're done. Otherwise, we continue in this fashion to produce a sequence x_n of points in this ball and a sequence ϵ_n of positive numbers such that

1. $\epsilon_n = \epsilon_{n-1}/2$
2. $\epsilon_n g(x_n) \geq \epsilon_0 g(x_0)$

Clearly, (1) says that $\epsilon_n \to 0$. Because X is complete, then the $x_n \to z \in X$. Claim: After finitely many steps, we should find a x_n and ϵ_n that are suitable.

Suppose this is false. Then $\epsilon_n = \epsilon_0/2^n$. Thus, $\epsilon_n g(x_n) = \epsilon_0/2^n g(x_n) \geq \epsilon_0 g(x_0)$ which means $g(x_n) \geq 2^n g(x_0)$. This implies that

$$\lim_{n\to\infty} \sup 2^n g(x_0) \leq \lim_{n\to\infty} \sup g(x_n) = g(z) < \infty.$$

This means that since 2^n grows exponentially, $g(x_0) = 0$. But g is a positive function. This contradiction gives us our result. $\square$

This lemma provides us with two new sequences that we still call by the same name: ϵ_k and (s_k, t_k) such that

$$\lim_{k\to\infty} \epsilon_k \|\nabla_{(s_k,t_k)} u_k\| = +\infty; \quad 2\|\nabla_{(s_k,t_k)} u_k\| \geq \|\nabla_{(s,t)} u_k\|, \text{ for } (s,t) \in B((s_k,t_k), \epsilon_k).$$

Recall that we use the notation $R_k = \|\nabla_{(s_k,t_k)} u_k\|$; then $\epsilon_k R_k \to \infty$. Define

$$v_k(s,t) = u_k\left(\frac{(s,t)}{R_k} + (s_k,t_k)\right)$$

so that, letting $(S_k, T_k) = (s,t)/R_k + (s_k,t_k)$

$$\|\nabla_{(s,t)} v_k\| = \frac{1}{R_k} \nabla_{(S_k,T_k)} u_k.$$

Then, for $(s,t) = (0,0)$, we have $\nabla_{(0,0)} v_k = \frac{1}{R_k} \nabla_{(s_k,t_k)} u_k$. By construction, $\|\nabla_{(0,0)} v_k\| = 1$. Moreover, on $B(0, \epsilon_k R_k)$,

$$\|\nabla_{(s,t)} v_k\| = \frac{1}{R_k} \|\nabla_{(S_k,T_k)} u_k\| \leq \frac{2}{R_k} \|\nabla_{(s_k,t_k)} u_k\| \leq 2,$$

meaning the gradient is uniformly bounded. Also, since the u_k are solutions, then the v_k satisfy the following:

$$\frac{\partial v_k}{\partial s} + J(v_k)\frac{\partial v_k}{\partial t} + \frac{1}{R_k} \nabla_{(t_k + t/R_k, v_k)} H = 0.$$

We may now apply an elliptic regularity lemma and extract a subsequence v_k which tend to a limit v in C^∞_{loc}. It will be a solution of the Floer equation and satisfy the following

- $\|\nabla_{(0,0)} v\| = 1$; so v is **not** constant
- $\|\nabla_{(s,t)} v\| \leq 2$ on $\mathbb{C}$
- $\frac{\partial v}{\partial s} + J(v)\frac{\partial v}{\partial t} = 0$; so v is J-holomorphic.

Through a series of inequalities, we're able to show that v has finite energy. What we see is that the gradient of v_k on a $B(0, \epsilon_k R_k)$ equals the gradient of u_k on the ball $B_k = B((s_k,t_k), \epsilon_k)$. As $k \to \infty$, the B_k are shrinking to a point and the energy of the u_k is blowing up.

On the other hand, when we choose this new parametrization with the v_k, we're recentering and recaling in such a way that we study instead, expanding balls, expanding to cover the whole of $\mathbb{R}^2$. The energies of the v_k's are bounded by some constant C, independent of k.

We now state three lemmas and prove two.

Lemma 4.3. *v has finite energy.*

Proof. Let $B_k = B((s_k, t_k), \epsilon_k)$. Then

$$\int_{B(0,\epsilon_k R_k)} |\nabla v_k| \, ds\, dt = \int_{B_k} |\nabla u_k| \, ds\, dt = \int_{B_k} \left(\left|\frac{\partial u_k}{\partial s}\right|^2 + \left|\frac{\partial u_k}{\partial t}\right|^2 \right) ds\, dt$$

Let $a = \partial u_k/\partial s$, $b = \partial u_k/\partial t$, $c = X_t$. Then the substituting these into the last integral and continuing, we find that:

$$\int |a|^2 + |b - c + c|^2 \leq \int |a^2| + |b-c|^2 + 2|b-c| \cdot |c| + |c|^2.$$

Observe that $0 \leq (|b-c| - |c|)^2 = |b-c|^2 - 2|b-c| \cdot |c| + |c|^2$. So $2|b-c| \cdot |c| \leq |b-c|^2 + |c|^2$. Substituting this into the above integrals, we now have

$$\begin{aligned} \int |a^2| + |b-c|^2 + 2|b-c| \cdot |c| + |c|^2 &\leq \int |a|^2 + 2|b-c|^2 + 2|c|^2 \\ &= \int_{B_k} \left|\frac{\partial u_k}{\partial s}\right| + 2\left|\frac{\partial u_k}{\partial t} - X_t(u_k)\right|^2 + 2|X_t(u_k)|^2 \\ &\leq 3E(u_k) + 2\int_{B_k} |X_t(u_k)|^2. \end{aligned}$$

Recall that the $E(u)$ are uniformly bounded by a constant $C > 0$ from above. Also, the $\epsilon_k \to 0$ so the B_k shrink to a point and thus, the last integral vanishes as $k \to \infty$. Hence, we can finally give a bound $\int_{B(0,\epsilon_k R_k)} |\nabla v_k| \leq 4C$ for all k. Since the $\epsilon_k R_k \to \infty$, in the limit, we have $\int_{\mathbb{R}^2} |\nabla v| \leq 4C$ by Fatou's lemma:

$$\int_{\mathbb{R}^2} |\nabla v| = \int_{\mathbb{R}^2} \lim_{k\to\infty} \inf |\nabla v_k| \leq \lim_{k\to\infty} \inf \int_{B(0,\epsilon_k R_k)} |\nabla v_k| \leq 4C.$$

Now, in the case of v, since it satisfies

$$\frac{\partial v}{\partial s} + J(v)\frac{\partial v}{\partial t} = 0,$$

then $|\nabla v|^2 = |\frac{\partial v}{\partial s}|^2 + |\frac{\partial v}{\partial t}|^2 = |\frac{\partial v}{\partial s}|^2 + |J\frac{\partial v}{\partial t}|^2 = 2|\frac{\partial v}{\partial s}|^2$. Thus, $2E(v) = \int_{\mathbb{R}^2} |\nabla v| \leq 4C$. So v has finite energy. □

Lemma 4.4 (6.6.4)**.** *The symplectic area of v is finite and positive.*

Proof.

$$\int_{\mathbb{R}^2} v^*\omega = \int_{\mathbb{R}^2} \omega\left(\frac{\partial v}{\partial s}, \frac{\partial v}{\partial t}\right) = \int_{\mathbb{R}^2} \omega\left(\frac{\partial v}{\partial s}, J(v)\frac{\partial v}{\partial s}\right) = \int_{\mathbb{R}^2} \left|\frac{\partial v}{\partial s}\right|^2 = E(v) \leq 2C.$$

But also, v can't have zero symplectic area because that would require $\partial v/\partial s = 0$ which then implies that $\partial v/\partial t = 0$. That would make v constant. But we saw from above that v is not constant. Hence, v has finite, positive symplectic area. □

Lemma 4.5 (6.6.5)**.** *There exists a sequence r_k tending to $+\infty$ such that the length of the image $v(\partial B(0, r_k))$ tends to 0 when k tends to $+\infty$.*

Using these two lemmas, we can obtain a contradiction. The 2nd lemma says that the image of the boundary of the ball is crushed to a point as $k \to \infty$. So for k large, the image is contained in a Darboux chart $U \subset M$. A closed form is locally exact; thus, in the Darboux chart, $\omega = d\lambda$. Let's suppose U is a closed ball and the curve $v(\partial B_r)$ is the boundary of a small disk $D_r \subset U$. The union of $v(B_r)$ with D_r gives a sphere S_r^2. We have this assumption that spheres have zero symplectic area:

$$0 = \int_{S_r^2} \omega = \int_{D_r} \omega + \int_{v(B_r)} \omega.$$

The first integral is

$$\left| \int_{D_r} \omega \right| = \left| \int_{D_r} d\lambda \right| = \left| \int_{v(\partial B_r)} \lambda \right| \leq \ell(v(\partial B_r)) \sup_U \|\lambda\|$$

where ℓ denotes length. The length goes to 0 as $r \to \infty$. The second integral converges to the area of $v(\mathbb{R}^2)$ which is nonzero. This is the contradiction we need $0 \neq 0$. Therefore, there couldn't be such sequences $u_k \in \mathcal{M}$ and $(s_k, t_k) \in \mathbb{R} \times S^1$ with $\|\nabla_{(s_k,t_k)} u_k\| \to \infty$. $\square$