Outline of Ch. 11: Floer Homology: Invariance

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We give an outline of ch. 11 of *Morse Theory and Floer Homology.*

1 Introduction

The main goal of this chapter is to show that the Floer homology does not depend on the choice of regular pair $(H, J) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$. Suppose we have (H^a, J^a) and (H^b, J^b) in $(\mathcal{H} \times \mathcal{J})_{\text{reg}}$.

Since we wish to show that both of these give rise to the same Floer homology, we consider a smooth homotopy $\Gamma(s)$ between these two, such that for some $R > 0$, when $s \leq -R$, the homotopy is stationary at (H^a, J^a) and when $s \geq R$, it is stationary at (H^b, J^b). This will produce for us, a chain morphism Φ^Γ between the two Floer complexes.

The proof of invariance has two main steps:

1. We define from this homotopy Γ, a morphism of complexes $\Phi^\Gamma : CF_*(H^a, J^a) \to CF_*(H^b, J^b)$. For a Γ which just stays on (H^a, J^a), we will show that Φ^Γ is the identity.

2. Suppose we have three pairs $(H^a, J^a), (H^b, J^b), (H^c, J^c)$ and homotopies $\Gamma, \Gamma', \Gamma''$ connecting a to c, a to b and b to c, respectively, then the morphisms

$$\Phi^{\Gamma''} \circ \Phi^{\Gamma'}, \Phi^\Gamma : CF_*(H^a, J^a) \to CF_*(H^c, J^c)$$

are chain homotopic and thus, induce the same morphisms on the homology level. Letting $(H^a, J^a) = (H^c, J^c)$ and $\Gamma = \text{id}$, then $\Phi^{\Gamma'}$ and $\Phi^{\Gamma''}$ are isomorphisms. This shows the invariance of Floer homology.

Therefore, we may take any Hamiltonian and form a homotopy to a C^2 small Hamiltonian and the Floer homology of the two will be isomorphic. In ch. 10, we saw that a C^2 small Hamiltonian gives us Floer homology which coincides with Morse homology and of course, Morse homology is isomorphic to singular homology.

Also, we have a lower bound on the number of periodic orbits: the sum of the dimension of the Floer homology groups. This lower bound comes about in the same way as in the Morse inequalities (the definition of the Floer complex is very similar to the Morse complex). From the prior discussion, the lower bound is in fact the sum of the Betti numbers. This proves the Arnold conjecture in our special case of symplectically aspherical manifolds.

2 The Morphism Φ^Γ

Let $\Gamma(s) = (H_{s,t}, J_s)$ and consider the equation

$$\mathcal{F}^\Gamma u = \frac{\partial u}{\partial s} + J_s(u)\frac{\partial u}{\partial t} + \nabla_u H_{s,t} = 0.$$

Though H and J now depend on s, this is **not** really a parametrized Floer equation since s was already a parameter. But it is a new Floer equation. When $s \leq -R$, the equation is the Floer equation with (H^a, J^a) and when $s \geq R$, it is the Floer equation with (H^b, J^b). We can once again consider the energy of a solution u and consider the moduli space of finite energy solutions $\mathcal{M}^\Gamma$.

It is our hope that such solutions connect criticals points but instead of having critical points of only one action functional, we have two: $\mathcal{A}_{H^a}, \mathcal{A}_{H^b}$.

Theorem 2.1 (11.1.1, paraphrased)**.** *For every $u \in \mathcal{M}^\Gamma$, there exists a critical point x of $\mathcal{A}_{H^a}$ and a critical point y of $\mathcal{A}_{H^b}$ such that $\lim_{s\to-\infty} u(s,\cdot) = x(\cdot)$ and $\lim_{s\to+\infty} u(s,\cdot) = y(\cdot)$.*

We also have the strong result that there is a constant $C > 0$ such that for any $u \in \mathcal{M}^\Gamma$, $E(u) < C$. This is enough to give us compactness:

Theorem 2.2 (11.1.4)**.** *$\mathcal{M}^\Gamma$ is compact in the C^∞_{loc} topology.*

Moreover, it turns out that all the solutions in $\mathcal{M}^\Gamma$ connect critical points of $\mathcal{A}_{H^a}$ to $\mathcal{A}_{H^b}$. There are no trajectories from critical points of $\mathcal{A}_{H^a}$ to other critical points of $\mathcal{A}_{H^a}$ as a result of the new Floer equation moving from a to b. The same holds for H^b.

Also, we can look for perturbations $h : \mathbb{R}\times S^1\times W \to \mathbb{R}$ to apply to our homotopy $\Gamma = (H, J)$ so that we obtain transversality and find that $\mathcal{M}^\Gamma$ is a smooth manifold. It turns out, this can be generically done when h is regular. Then, given any pair of critical points x and y of $\mathcal{A}_{H^a}$ and $\mathcal{A}_{H^b}$ respectively, the dimension of $\mathcal{M}^\Gamma(x, y)$ is computed by considering the Fredholm index of $d\mathcal{F}$ where $\mathcal{F}$ is the Floer map for the new Floer equation. The index is $\mu(x) - \mu(y)$. This is a nice result as, remember, x and y are critical points for **different** action functionals.

However, **we do not quotient by** $\mathbb{R}$ because a solution to $\mathcal{F}^\Gamma$, when translated, may no longer be a solution. If we take a sequence $u_n \in \mathcal{M}^\Gamma$ and translate by a sequence $s_n \to +\infty$, then we could make it so that u_n converges to something in $\mathcal{M}_b$ which is then not an element of $\mathcal{M}^\Gamma$. This does not contradict the compactness of $\mathcal{M}^\Gamma$ because here, we're trying to do something extra: translate the sequence.

The definition of $\Phi^\Gamma : CF_*(H^a, J^a) \to CF_*(H^b, J^b)$ is as follows: map $x \in \mathrm{Crit}_k(\mathcal{A}_{H_a})$ to

$$\Phi^\gamma(x) = \sum_{y\in \mathrm{Crit}_k(A_{H^b})} n^\Gamma(x, y)y.$$

To prove that this is a chain morphism, we need to prove:

1. $n^\Gamma(x, y)$ which is the number of trajectories connecting x and y (same index but for different action functionals) taken mod 2, is **finite**. This requires compactness of $\mathcal{M}^\Gamma(x, y)$ and some regularity.

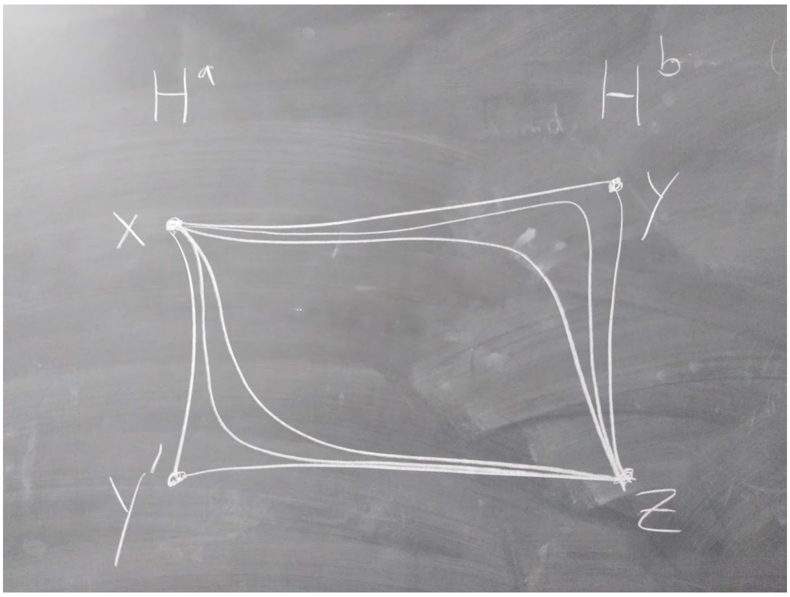
2. We need to show that $\partial^b \circ \Phi^\Gamma = \Phi^\Gamma \circ \partial^a$. Since everything is taken mod 2, we can instead show that $\partial^b \circ \Phi^\Gamma + \Phi^\Gamma \circ \partial^a = 0$; we need the sum of these to be even.

See the picture which illustrates what is going on.

In the picture, the horizontal trajectories between x and y and y' and z are counted by Φ^Γ while the vertical ones are counted by ∂. We also see trajectories converging to broken trajectories. This is, of course, an artifact of how we're presenting the geometry; there's no mathematical notion of a "horizontal trajectory" other than one that connects critical points of the same index on the a and b side.

For (2), as stated above, this amounts to showing that

$$\sum n^b(x, y')n^\Gamma(y'z) + \sum n^\Gamma(x, y)n^a(y, z) = 0 \pmod 2 \tag{2.1}$$



Geometry Behind the Chain Complex Algebra

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We have two terms here. What does it mean geometrically? The fact that there are two terms suggests that the trajectories break in only two ways: either to the upper right or lower left in the picture. *A priori*, this is not clear; potentially, it seems there are many more ways to break with critical points of higher or lower index.

However, the geometry prevents such behavior in the form of the following compactness theorem:

Theorem 2.3 (11.1.10, paraphrased)**.** *Let $(u_n) \in \mathcal{M}^\Gamma(x, y)$ be a sequence. There exists a subsequence, still denoted u_n and*

- *Critical points $x = x_0, x_1, ..., x_k$ of $\mathcal{A}_{H^a}$;*
- *Critical points $y_0, y_1, ..., y_\ell = y$ of $\mathcal{A}_{H^b}$;*
- *Real sequences s_n^i for $0 \le i \le k-1$ that tend to $-\infty$ and t_n^j for $0 \le j \le \ell - 1$ that tend to $+\infty$;*
- *Elements $u^i \in \mathcal{M}_a(x_i, x_{i+1})$ and $v^j \in \mathcal{M}_b(y_j, y_{j+1})$ for the i and j as above;*
- *An element $w \in \mathcal{M}^\Gamma(x_k, y_0)$;*

such that for the i's and j's,

$$\lim_{n\to+\infty} u_n \cdot s_n^i = u^i, \qquad \lim_{n\to+\infty} u_n \cdot t_n^j = v^j$$

and $\lim_{n\to+\infty} u_n = w$*.*

This theorem tells us how trajectories from x to y can break. Moreover, this theorem and also **transversality** give us a corollary: If we have a broken trajectory where it breaks k times on the H^a side and ℓ times on the H^b side, there is an extremely useful inequality: $\mu(x) - \mu(y) \ge k + \ell$. We have this because by transversality, all other breaks will come from moduli spaces of negative formal dimension which are empty.

So in the case that $\mu(x) = \mu(y)$, there is no breaking. In the case that $\mu(x) = \mu(z) + 1$, there are only two ways to break: $k = 1, \ell = 0$ or $k = 0, \ell = 1$. Compare this to when we're looking at only one (H, J); then $\mu(x) - \mu(y) > \#$ number of breaks; note the **strict** inequality. For $\mathcal{M}^\Gamma$, we have $\geq$ because we have one extra dimension of room from lack of quotienting by $\mathbb{R}$.

Before we continue to prove (2), let us state how we can prove (1), now that we have this inequality. We need to show that $\mathcal{M}^\Gamma(x, y)$ is compact. Let u_n be a sequence in here. The inequality we just saw shows that when $\mu(x) = \mu(y)$, there are no breaks in the trajectories. So $x = x_k$ and $y = y_0$ as in Theorem 11.1.10. But the theorem also tells us that $u_n \to w \in \mathcal{M}^\Gamma(x, y)$, hence we have sequential compactness.

Lastly, since $\dim \mathcal{M}^\Gamma(x, y) = 0$, this shows that we only have a finite number of trajectories to count and hence, Φ^Γ is well-defined.

Returning our attention back to proving (2), in order to verify equation 2.1, it suffices to verify that the number of points of the compact manifold of dimension 0

$$\Pi^\Gamma(x, z) = \bigcup_{\substack{y' \in \mathrm{Crit}\mathcal{A}_{H^a} \\ \mu(x)-\mu(y')=1}} \mathcal{L}_a(x, y') \times \mathcal{M}^\Gamma(y', z) \cup \bigcup_{\substack{y \in \mathrm{Crit}\mathcal{A}_{H^b} \\ \mu(x)=\mu(y')}} \mathcal{M}^\Gamma(x, y) \times \mathcal{L}_b(y, z)$$

is even. This is a consequence of the following theorem:

Theorem 2.4. *For $x \in Crit\mathcal{A}_{H^a}$ and $z \in Crit\mathcal{A}_{H^b}$ with $\mu(x) - \mu(z) = 1$, the space $\mathcal{M}^\Gamma(x, z) \cup \Pi^\Gamma(x, z)$ is a compact manifold of dimension 1 with boundary, and its boundary is $\Pi^\Gamma(x, z)$.*

By theorem 11.1.6, we know that $\mathcal{M}^\Gamma(x, z)$ is a 1-manifold without boundary. Using convergence toward broken trajectories, we can define a topology on $\mathcal{M}^\Gamma(x, z) \cup \Pi^\Gamma(x, z)$ that is compatible with $\mathcal{M}^\Gamma(x, z)$ (see the picture above). This topology is Hausdorff and compact. But we need to show that it is indeed a manifold with boundary. Thus, we need a gluing theorem.

3 Gluing

Previously, we had a gluing theorem to show that the moduli space for the standard Floer equation is a manifold with boundary of some dimension. Prior to gluing, we were able to establish that the moduli space is compact and away from the boundary, is a manifold. The difficulty was in knowing how to "glue" the boundary onto this moduli space.

The current situation is much the same but we're considering the modified Floer equation which involves both (H^a, J^a) and (H^b, J^b). Here is the theorem:

Theorem 3.1 (11.1.16, Gluing)**.** *Let x be a critical point of $\mathcal{A}_{H^a}$ and let y, z be critical points of $\mathcal{A}_{H^b}$ such that $\mu(x) = \mu(y) = \mu(z) + 1$. Let $u \in \mathcal{M}^\Gamma(x, y)$ and let $\hat{v} \in \mathcal{L}_{(H^b, J^b)}(y, z)$. Then:*

- *There exists an embedding $\psi : [\rho_0, +\infty) \to \mathcal{M}^\Gamma(x, z)$ (for some $\rho_0 > 0$) such that $\lim_{\rho \to +\infty} \psi_\rho = (u, \hat{v})$.*
- *Moreover, if ℓ_n is a sequence of elements of $\mathcal{M}^\Gamma(x, z)$ that tends to $(u, \hat{v})$, then $\ell_n \in Im\,\psi$ for n sufficiently large.*

The steps to proving this theorem are quite similar to the previous gluing theorem:

1. Define a pregluing w_ρ for $u \in \mathcal{M}^\Gamma(x, y)$ and $v \in \mathcal{M}_b(y, z)$ (not $\hat{v}$). It uses bump functions and exponential maps to make it so that $w_\rho = u$ for $s \leq -1$ and $w_\rho = v$ for $s \geq 1$.
2. Consider an operator
$$\mathcal{F}_\rho^\Gamma = \frac{\partial}{\partial s} + J_{s+\rho} \frac{\partial}{\partial t} + \nabla H_{s+\rho, t}.$$
Then, $\mathcal{F}_\rho^\Gamma(w_\rho)(s, t) = 0$ for $|s| \geq 1$.

3. Define a nonlinear operator $F_\rho^\Gamma : B(0,r) \subset W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ by considering unitary trivializations (the r here is some small positive constant dependent on the exponential maps injective radius and some other quantity). If $\xi = \sum y^i Z_i^\rho$, then $F_\rho^\Gamma(y_1, ..., y_{2n}) = \mathcal{F}_\rho^\Gamma(\exp_{w_\rho} \xi)$. Observe that $F_\rho^\Gamma(0) = 0$ for $|s| \geq 1$ (hence the nonlinearity).

4. Let $W_\rho = \{0\#_\rho\beta : \beta \in \ker(dF^v)_0\}$. Since we have an approximate solution for a zero of F_ρ^Γ, we may apply the Newton-Picard method to F_ρ^Γ to obtain the following result: there exists a $\rho_0 > 0$ such that for all $\rho \geq \rho_0$, there exists a $\gamma(\rho) \in W_\rho^\perp$ with $F_\rho(\gamma(\rho)) = 0$ (everywhere). γ is unique in $B(0,\epsilon) \cap W_\rho^\perp$ and has properties we need in order to define our gluing embedding ψ.

5. Define $\varphi_\rho = \exp_{w_\rho} \gamma(\rho)$ and $\psi_\rho(s,t) = \varphi_\rho(s-\rho, t)$. This ψ has all the properties claimed in the theorem.

All this work finally gives us a moduli space which is a 1-manifold **with boundary**. The chain map Φ^Γ is well-defined and indeed a chain morphism. That is, the number of elements of $\mathcal{M}^\Gamma(x,y)$ is finite so that $n^\Gamma(x,y)$ gives a mod 2 count of the trajectories.

4 Invariance with respect to Γ

Our work so far has been to show that the Floer homology is independent of choice of regular time-dependent Hamiltonian and almost complex structure. To show this, we chose a regular homotopy Γ connecting (H^a, J^a) and (H^b, J^b). However, what if this depends on our choice of Γ? We need to show that the choice of Γ does not matter and thus, we take a homotopy of homotopies. Let's say we have homotopies Γ_0, Γ_1. We want to show they induce morphisms Φ^{Γ_0} and Φ^{Γ_1} which coincide on the homology level. This amounts to showing that there exists a map $S : CF_{*+1}(H^a, J^a) \to CF_*(H^b, J^b)$ which satisfies $\Phi^{\Gamma_0} - \Phi^{\Gamma_1} = S \circ \partial_a + \partial_b \circ S$. Taken mod 2, we can rewrite this as $\Phi^{\Gamma_0} + \Phi^{\Gamma_1} + S \circ \partial_a + \partial_b \circ S = 0$.

Let us define a homotopy (of homotopies) between Γ_0 and Γ_1 by $\Lambda = (\Gamma_\lambda)_{\lambda \in [0,1]}$. We'll define a Φ^Λ which will be our map S. We have the following situation with chain complexes:

$$\begin{tikzcd}
\dots \arrow[r, "\partial^a"] & CF_{k+1}(H^a, J^a) \arrow[r, "\partial^a"] \arrow[d, "\Phi^{\Gamma_0}, \Phi^{\Gamma_1}"'] & CF_k(H^a, J^a) \arrow[r, "\partial^a"] \arrow[d, "\Phi^{\Gamma_0}, \Phi^{\Gamma_1}"'] \arrow[dl, "S"'] & CF_{k-1}(H^a, J^a) \arrow[r, "\partial^a"] \arrow[d, "\Phi^{\Gamma_0}, \Phi^{\Gamma_1}"'] \arrow[dl, "S"'] & \dots \\
\dots \arrow[r, "\partial^b"] & CF_{k+1}(H^b, J^b) \arrow[r, "\partial^b"] & CF_k(H^b, J^b) \arrow[r, "\partial^b"] & CF_{k-1}(H^b, J^b) \arrow[r, "\partial^b"] & \dots
\end{tikzcd}$$

To reiterate, hidden in here is the fact that we're using a Λ which is parametrized by λ, unlike our previous situation where Γ was a fixed homotopy. To show invariance in this setting, we need to show that

$$\sum n^b(x,y')n^\Lambda(y'z) + \sum n^\Lambda(x,y)n^a(y,z) + \sum n^{\Gamma_0}(x,z) + \sum n^{\Gamma_1}(x,z) = 0 \pmod 2.$$

These four terms correspond to the equation $\partial^b \circ \Phi^\Gamma + \Phi^\Gamma \circ \partial^a + \Phi^{\Gamma_0} + \Phi^{\Gamma_1} = 0$
An Important Interjection: Consider the following equations regarding chain complexes that we've seen:

- $\partial^2 = 0$
- $\partial^b \circ \Phi^\Gamma + \Phi^\Gamma \circ \partial^a = 0$

- $\partial^b \circ \Phi^\Lambda + \Phi^\Lambda \circ \partial^a + \Phi^{\Gamma_0} + \Phi^{\Gamma_1} = 0$

To prove each of them, we need to count broken trajectories. Thus, the geometry really is giving rise to the algebra.

Now, one may expect that having a homotopy of homotopies will give rise to a family of moduli spaces $\mathcal{M}^{\Gamma(\lambda)} \to [0,1]$ which forms a cobordism between $\mathcal{M}^{\Gamma_0}$ and $\mathcal{M}^{\Gamma_1}$. This is a good thought and is "almost" true. cf. Seiberg-Witten theory and wall-crossing. However, the homotopy of homotopies might not pass through only regular homotopies. Thus, as we move along $[0,1]$, we get additional types of boundary. However, as we'll see below, the four terms always give an even integer.

Another concern is that one might wish to ask if the choice of homotopy of homotopies matters. But the goal is just to show that Φ^{Γ_0} and Φ^{Γ_1} induce the same morphisms on homology; it doesn't matter if the choice of homotopy of homotopies makes a difference on the chain level.

Now, back to showing invariance: we want to define Φ^Γ where Γ is a homotopy of homotopies and satisfies

$$\partial^b \circ \Phi^\Lambda + \Phi^\Lambda \circ \partial^a + \Phi^{\Gamma_0} + \Phi^{\Gamma_1} = 0. \tag{4.1}$$

Let us consider the **family of Floer equations** parametrized by λ; this time, λ is a brand new parameter so this is a **true** family of equations, unlike last time when we just introduced s into H and J:

$$\frac{\partial u}{\partial s} + J_s^\lambda(u)\frac{\partial u}{\partial t} + \nabla_u H_{s,t}^\lambda = 0.$$

Fix a $\lambda \in [0,1]$ and let $\mathcal{M}^{\Gamma_\lambda}$ be solutions of the parametrized Floer equation with finite energy. Let x be a critical point of $\mathcal{A}_{H^a}$ and y a critical point of $\mathcal{A}_{H^a}$. Then $\mathcal{M}^{\Gamma_\lambda}(x,y)$ will be defined as expected and $\mathcal{M}^\Lambda(x,y) = \{(\lambda, u) : u \in \mathcal{M}^{\Gamma_\lambda}(x,y)\}$. We also have a similar inequality as before. If k is the number of breaking points in a trajectory on the a side and ℓ is the number of breaking points on the b side, then $\mu(x) - \mu(y) + 1 \geq k + \ell$.

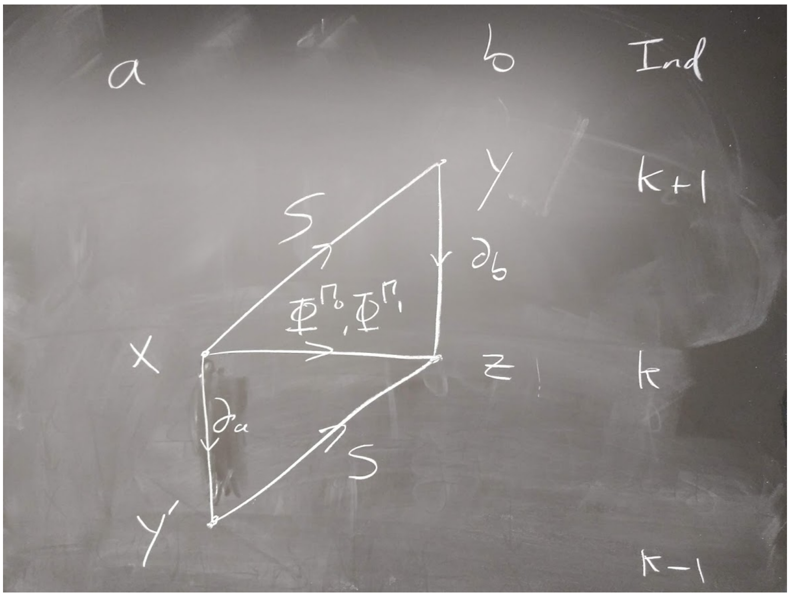
Let x be a critical point of $\mathcal{A}_{H^a}$ of index i. For the sake of consistency with the book, let us define $S := \Phi^\Lambda$ by $S_i(x) = \sum m^\Lambda(x,y)y$ where these y range over all critical points y of $\mathcal{A}_{H^b}$ with $\mu(y) = i+1$. Of course, the dimension of $\mathcal{M}^\Lambda(x,y)$ is $\mu(x) - \mu(y) + 1 = i - (i+1) + 1 = 0$. Then $m^\Lambda(x,y)$ is the number of points of $\mathcal{M}^\Lambda(x,y)$ taken mod 2.

Now, let $x, y' \in \operatorname{Crit}(\mathcal{A}_{H^a})$ and $z, y \in \operatorname{Crit}(\mathcal{A}_{H^b})$. x and z will be fixed critical points but y and y' may vary. We'll have $\mu(x) = \mu(z) = \mu(y') + 1 = \mu(y) - 1$. Define

$$\Pi^\Lambda(x,z) = \left(\bigcup_{y'} \mathcal{L}_a(x,y') \times \mathcal{M}^\Lambda(y',z)\right) \cup \left(\bigcup_{y} \mathcal{M}^\Lambda(x,y) \times \mathcal{L}_b(y,z)\right).$$

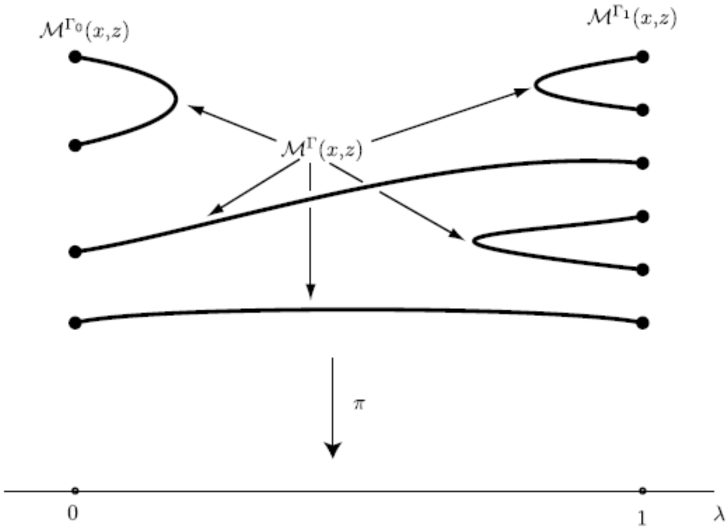
This is what we might naturally expect to be the boundary of $\mathcal{M}^\Lambda(x,z)$ (which is an open 1-manifold because $\mu(x) = \mu(z)$); however, this is **not true** unless our homotopy always passes through regular homotopies. To show that $S := \Phi^\Lambda$ satisfies equation 4.1, we need to show, essentially, that $\mathcal{M}^\Lambda(x,z) \cup \Pi^\Lambda(x,z)$ is a 1-manifold with boundary. The terms $\partial^b \circ \Phi^\Gamma + \Phi^\Gamma \circ \partial^a$ in equation 4.1 come from $\Pi^\Lambda(x,z)$ while the other two terms come from $\mathcal{M}^\Lambda(x,z)$. The pictures below suggest that $\Pi^\Lambda(x,z)$ always comes with an even number of points. This is a **misleading** feature of the pictures. In general, it is arbitrary how the points pair up and add to zero.

Of course, to show that $\mathcal{M}^\Lambda(x,z) \cup \Pi^\Lambda(x,z)$ is a 1-manifold with boundary is a **gluing** problem. The techniques are similar as before, such as using Newton-Picard. But we now must account for the parametrizations. Here is a picture which shows how the trajectories can break; but the picture is drawn in a way to resemble what we're seeing on the chain level.



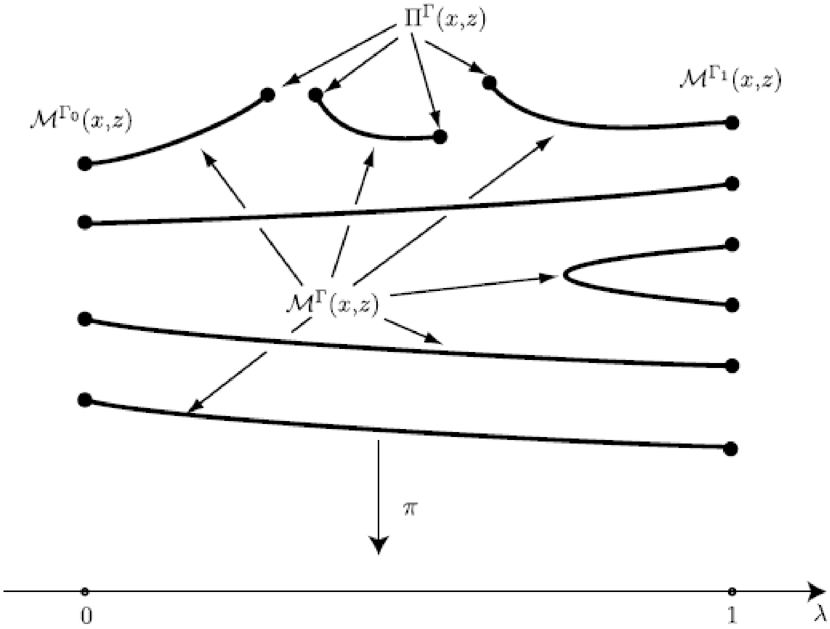
The geometry of broken trajectories reflects the algebra on the chain level

Note: in the case that $\Gamma(\lambda)$ is a regular homotopy for each $\lambda \in [0,1]$, then in fact, $\Pi^{\Lambda}(x,z)$ is **empty**. In this case, $\mathcal{M}^{\Lambda}(x,z)$ is a cobordism between $\mathcal{M}^{\Gamma_0}(x,z)$ and $\mathcal{M}^{\Gamma_1}(x,z)$. Here is a figure from Audin and Damian.



When each Γ_λ is a regular homotopy, we have a cobordism (the picture uses Γ instead of Λ)

However, if there are some $\Gamma(\lambda)$ which are not regular homotopies for $\lambda \in (0,1)$, then we get something almost like a cobordism but will have some points coming from $\Pi^{\Lambda}(x,z)$ which is **nonempty**. This is expected because we're looking at a family of Floer equations parametrized by λ; we might run into non-regular pairs (H,J) because our generic condition only guarantees that the non-regular pairs live in some codim 1 space; a path might cross the codim 1 wall (cf. Seiberg-Witten theory). Here is a figure:



When some Γ^λ is not regular, $\Pi^\Gamma(x, z)$ is nonempty (the picture uses Γ instead of Λ)

Let us suppose that this gluing can be done and that we've established equation 4.1. Thus, we only have two steps left.

Let $\Gamma' = (H', J')$ link (H^a, J^a) to (H^b, J^b) and $\Gamma'' = (H'', J'')$ link (H^b, J^b) to (H^c, J^c). We may concatenate Γ' and Γ'' to form $\Gamma_\rho = (H_\rho, J_\rho)$. Here,

$$H_\rho(s, t, \rho) = \begin{cases} H'(s + \rho, t, \rho), & s \leq 0 \\ H''(s - \rho, t, \rho), & s \geq 0 \end{cases}$$

$$J_\rho(s, \rho) = \begin{cases} J'(s + \rho, \rho), & s \leq 0 \\ J''(s - \rho, \rho), & s \geq 0. \end{cases}$$

Lemma 4.1. *Γ_ρ might not be regular but by perturbing Γ' and Γ''. After doing so, Γ_ρ is regular for large ρ.*

Proposition 4.2. *There exists $\rho > 0$ such that Γ_ρ is regular and the morphism of complexes $\Phi^{\Gamma'} \circ \Phi^{\Gamma''} = \Phi^{\Gamma_\rho}$. Thus, they coincide on the homology level.*

Let us summarize our results (the last one needs a bit of proof):

- On homology, Φ^Γ induces a morphism independent of the homotopy Γ between (H^a, J^a) and (H^b, J^b).
- If Γ' and Γ'' are as above, linking regular pairs, then there exists Γ, linking (H^a, J^a) and (H^c, J^c), such that $\Phi^{\Gamma'} \circ \Phi^{\Gamma''} = \Phi^{\Gamma_\rho}$ on homology.
- When $(H^a, J^a) = (H^c, J^c)$ and $\Gamma = \mathrm{id}$, then $\Phi^{\mathrm{id}} = \mathrm{id}$. Proof: The Floer equation here is just the regular one as Γ is stationary. There is only one trajectory from a critical point x to x, namely x itself is a trajectory (it satisfies the Floer equation and has finite energy equal to 0). Thus, $\Phi^{\mathrm{id}}(x) = x$.

With these results, $\Phi^{\Gamma'} \circ \Phi^{\Gamma''} = \mathrm{id}$ on homology and so Floer homology is invariant under choice of regular pairs (H, J). This proves Arnold's conjecture in our symplectically aspherical case because we can choose an autonomous C^2 small Hamiltonian H.