A Simple Example of the Atiyah-Singer Index Theorem

Sam Auyeung

January 27, 2020

The purpose of this note is to give one of the simplest examples of the Atiyah-Singer index theorem that I know of: the index of the operator $d + d^*$ on a closed Riemann surface Σ_g is $2g-2$ which is minus the Euler characteristic of Σ_g. I mostly just make use of a few facts from Hodge theory.

Let $D = d + d^* : \Omega^1(\Sigma_g, \mathbb{R}) \to \Omega^2(\Sigma_g, \mathbb{R}) \oplus \Omega^0(\Sigma_g, \mathbb{R})$ be an operator defined on differential forms of Σ_g. Recall that the Hodge theorem tells us that for k-forms on a closed, orientable smooth manifold M, we have a decomposition: $\Omega^k = \mathcal{H}^k(M) \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*$. Here, $\mathcal{H}^k(M)$ refers to the space of harmonic k-forms on M. That is, letting $\Delta = dd^* + d^*d$, a k-form α is harmonic iff $\Delta\alpha = 0$.

I claim that D is an elliptic operator and thus, is Fredholm (because the underlying manifold is compact). Then, the Fredholm index of D is by definition $\dim\ker D - \dim\operatorname{coker} D$. First, it is not too hard to see that since $\dim \Sigma_g = 2$, then $\Omega^0 = \mathcal{H}^0 \oplus \operatorname{Im} d^*$ and $\Omega^2 = \mathcal{H}^0 \oplus \operatorname{Im} d$. This is because we only have nontrivial forms in degree 0, 1, and 2.

Next, we recall that $\dim \mathcal{H}^k = \dim H^k$; i.e. the dimension of the space of harmonic k-forms is the same as the dimension of the kth de Rham cohomology. Since the cokernel of d is simply $\Omega^2/\operatorname{Im} d$, this is just $\mathcal{H}^2$ which is one dimensional. Similarly, this holds for the cokernel of d^*. Thus, $\dim\operatorname{coker} D = 2$.

Next, we also recall that $\Delta\alpha = 0$ iff $d\alpha = d^*\alpha = 0$. Moreover, $d^2 = (d^*)^2 = 0$. Thus, we see that the kernel of D should at least contain $(H)^1$. But while $d\alpha \in \operatorname{Im} d$ is in the kernel of d, is it the case that $d^*d\alpha = 0$? Thus, let's suppose that $h + d\alpha + d^*\beta \in \ker D$. This means that $D(h + d\alpha + d^*\beta) = d^*d\alpha + dd^*\beta = 0$. Now, α is a 0-form while β is a 2-form. This means that $d^*d\alpha$ is also a 0-form while $dd^*\beta$ is a 2-form. Thus, they cannot "cancel" each other in any way. Instead, they must each individually equal 0.

So what we now know is that $d^*d\alpha = 0$ and $dd^*\beta = 0$. But we also know that $d^2\alpha = (d^*)^2\beta = 0$. Hence, $d\alpha$ and $d^*\beta$ are both harmonic 1-forms. On the other hand, they're suppose to be orthogonal to $\mathcal{H}^1$ and thus, must both be equal to 0 to begin with. Therefore, the kernel of D is in fact, just $\mathcal{H}^1$ and $\dim\ker D = 2g$.

Therefore, $\operatorname{Ind} D = 2g - 2 = -\chi(\Sigma_g)$.