A Mathematical Viewpoint of Classical and Quantum Mechanics

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This is a set of notes which I hope will be illuminating when it comes to broader concepts but will not contain many details. It was compiled in the Spring 2020 semester for the course Mathematical Physics II taught by Leon Takhtajan.

1 Introduction

I would like to discuss classical and quantum mechanics from a rather general mathematical framework. The basic mantra will be, "Classical mechanics is mainly measure theory and quantum mechanics is mainly operator theory (functional analysis)."

In physics, we typically care about at least three things: states, observables, and dynamics. Physically, one might think of observables as things we can measure, states as being those a set of variables describing a system which does not include anything about its history, and dynamics is about how the system evolves over time. We'll see that these three notions appear in both CM and QM. And thus, it's not so hard to mathematically translate from one to the other once we have a basic framework.

2 Classical Mechanics

The general setting of classical mechanics takes place on a **phase space** which we'll take to be a **Poisson manifold** $(M, \{,\})$. We'll also have a **commutative algebra of observables** $A = C^\infty(M, \mathbb{R})$, and a **set of states** $S = \mathcal{P}(M)$ which are all probability measures on M.

Recall that the Poisson bracket we care about works with A: $\{,\} : A \otimes A \to A$. It satisfies:

1. Anticommutativity: $\{g, f\} = -\{f, g\}$.
2. Bilinearity: $\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \{h, af + bg\} = a\{h, f\} + b\{h, g\}$.
3. Leibniz Rule: $\{fg, h\} = \{f, h\}g + f\{g, h\}$.
4. Jacobi Identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

Also, a symplectic form ω on M would induce a Poisson bracket. The Poisson bracket can also be used to define Hamiltonian vector fields: $X_H(f) = \{H, f\}$. Supposing we do have a symplectic form ω, if the vector field gives us flows g_t, we can see that g_t preserves ω if and only if $\{,\}$ satisfies the Jacobi identity. Thus, in this way, the Jacobi identity has an interpretation. Recall that if the manifold is compact, then we do have existence of flows.

Now that we have this general setting, how do we study a specific physical system? The physical system is determined by choosing a Hamiltonian function $H : M \to \mathbb{R}$. This is enough

to completely determine the behavior of the system once we set the equations of motion; i.e. once we set the dynamics.

But before we get to that, what is a measurement? From physics, this is intuitive but mathematically, it is a map $A \times S \to \mathcal{P}(\mathbb{R})$ (probability measures on $\mathbb{R}$). The map is easily defined: $(f, \mu) \mapsto \mu_f$ where if $E \subset \mathbb{R}$ is a Borel set, then $\mu_f(E) := \mu(f^{-1}(E))$.

The expected value is $E_{\mu_f} = \int_{\mathbb{R}} \lambda \, d\mu_f(\lambda)$. By Fubini's theorem, this equals $\int_M f d\mu$. Now, the Dirac delta measures are called **pure state** and all other states are called **mixed states**. It is a fact that μ is a pure state if and only if the variance of μ_f is zero, $\forall f \in A$. This makes sense: classically, there should be no fuzzy uncertainty in our measurements, unlike in quantum mechanics. Note that for a pure state, the expected value is simply the value of the function at the point on which the measure is supported.

Now let's turn to the dynamics. It turns out there are two equivalent formulations of the dynamics: a Hamiltonian and Liouville formulation.

For the Hamiltonian situation, we have two equations:

$$\frac{d\mu}{dt} = 0, \quad \frac{df}{dt} = \{H, f\}.$$

The first equation tells us that the measure μ is independent of time.

Alternatively, if we assume Hamiltons euqations are solvable for all time, we can define a 1-paramter group of diffeomorphisms on M: g_t. Additionally, if we're working with a symplectic manifold (M, ω), then ω^n gives us volume measure $dx = \omega^n$. By Radon-Nikodym, $d\mu = \rho \, dx$ for some ρ. Our equations of motion are therefore:

$$\frac{df}{dt} = 0, \quad \frac{d\rho}{dt} = -\{H, \rho\}.$$

3 Quantum Mechanics

Now that we've seen the setting for classical mechanics, let's consider the following theorem:

Theorem 3.1 (Gelfand)**.** *"Every" commutative (Banach) $\mathbb{R}$-algebra is an algebra of continuous functions on some compact topological space.*

Thus, barring the technicalities of saying what we mean by "every," we see that in some sense, if our algebra is commutative, we're already in the realm of classical mechanics. This hints to us that in order to do quantum mechanics, we should look at **noncommutative algebras**. Since in the classical case, our A is ∞-dimensional, we ought to consider ∞-dimensional noncommutative algebras.

On first attempt, we might consider something like $\mathcal{A} = \mathcal{L}(\mathcal{H})$, the space of bounded operators on a Hilbert space $\mathcal{H}$, over $\mathbb{C}$. But we'll quickly see, by once again considering the classical picture, that we'll want **unbounded** operators. For consider the simplest phase space: $T^*\mathbb{R}$ with coordinates (p, q). These coordinates are unbounded. So we might want to take $\mathcal{A}$ to be the space of all operators, including unbounded ones, on a Hilbert space $\mathcal{H}$. But we'll find later that we will, once again, want to revise what counts as our observables.

Let us turn our attention towards defining something like our Poisson bracket, namely a Lie bracket. Take $A, B \in \mathcal{A}$ and define $[A, B] := AB - BA$, the commutator. This $[,]$ satisfies the Jacobi identity if and only if the operator product is associative. So this tells us that associativity is actually quite fundamental even though we often take it for granted when doing algebra!

3.1 Observables

From here, I will try to present what's called the Dirac-von Neumann axioms of quantum mechanics. We already have some view of what we might want $\mathcal{A}$ to be. But we'll see that we actually want another condition on our operators. First, let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$; we'll assume this is given to us (see the "Quantization" section below for a bit more on the origins of $\mathcal{H}$). Let $\mathcal{A}$ be the space of self-adjoint operators on $\mathcal{H}$; this includes unbounded operators. This is an "algebra" of quantum observables.

However, it's not really an algebra! If A and B are self-adjoint, $(AB)^* = B^*A^* = BA$. Thus, the product AB is self-adjoint if and only if A and B commute; this obviously doesn't always happen. In fact, what does it even mean to commute? These operators are unbounded! We'll discuss this a bit more later but for now, to address this issue of it not being an algebra, we can define a **Lie bracket**: $[A, B] := i(AB - BA)$. Then, if A, B are self-adjoint, $[A, B]^* = -i((AB)^* - (BA)^*) = -i(BA - AB) = i(AB - BA) = [A, B]$. We can also define a Poisson bracket $\{A, B\}_\hbar = \frac{i}{\hbar}(AB - BA)$ called the quantum bracket. So we have a Lie algebra.

3.2 States

We'll let $\mathcal{S}$ be the space of compact, positive, trace-class operators with trace equal to 1. As it turns out, the notion of self-adjointness is rather tricky in ∞-dimensions. For example, the unbounded operator $A = i\frac{d}{dx}$ is not defined on all of $L^2(\mathbb{R})$ but is defined on a dense subset. However, we do find it to be self-adjoint once we make clear what self-adjointness means.

A natural question to ask now is: "Why do we consider self-adjoint operators?" The answer is simply that we have von Neumann's spectral theorem. Essentially, it tells us that the self-adjoint operators on a Hilbert space are in 1-1 correspondence with projection-valued measures defined on the Borel sets of $\mathbb{R}$. A projection $P : \mathcal{H} \to \mathcal{H}$ is a bounded operator satisfying $P^2 = P, P = P^*$. A projection-valued measure simply gives us a projection instead of a real number when we input a Borel set.

This is how we go about assigning a self-adjoint operator A a projection-valued measure P_A. Let λ_i be the eigenvalues (with multiplicity) of A and V_i the corresponding eigenspaces. Let $P_{V_i} : H \to H$ be projection onto V_i. Then $A = \sum \lambda_i P_{V_i}$. We can then let $P_A(E) = \sum_{\lambda_i \in E} P_{V_i}$.

But the theorem doesn't stop here. It gives us criterion for determining whether f is in the domain of A. It turns out that $f \in D(A)$ if and only if

$$\int_{\mathbb{R}} \lambda^2 \, d(P_A(\lambda)f, f) < \infty.$$

This condition basically tells us that there is some L^2 thing to check. Moreover, von Neumann showed that once we know $f \in D(A)$, we know that

$$Af = \int_{\mathbb{R}} \lambda \, dP_A(\lambda) f.$$

Moving on, a compact operator is a bounded operator which maps the unit ball to a compact set. A is a positive operator if $(Af, f) \geq 0$ for all $f \in H$. This implies that A is self adjoint because, being a real value, $(f, Af) = \overline{(Af, f)} = (Af, f)$. It may be enough to check this on eigenvectors because by the Hilbert-Schmidt decomposition (see the next bit), we can describe A.

To discuss what it means to be trace-class will require a bit more discussion. Suppose we have a compact, self adjoint operator A. The Hilbert-Schmidt decomposition theorem tells us that we can decompose A by projections:

$$A = \sum_{n=1}^{N} \lambda_n P_n$$

where N can equal ∞. The theorem also tells us that A has a pure, discrete spectrum. This means that it has at most countably many non-zero eigenvalues of finite multiplicity; moreover, they should accumulate at 0. On the other hand, we do allow for the possibility that 0 has infinite multiplicity as an eigenvalue.

Now, A is **trace-class** if $\sum |\lambda_n| < \infty$. We define the trace of A to be $\sum^N \lambda_n$. It is nontrivial result that this trace can be described a different way. For any orthonormal basis $\{e_i\}^\infty$, we have

$$\operatorname{tr} A = \sum^{\infty} (Ae_i, e_i).$$

This relates a spectral version of trace with a "matrix version." It turns out that the space of trace-class operators is a 2-sided ideal in the Banach algebra of bounded operators on $\mathcal{H}$: $\mathcal{L}(\mathcal{H})$.

3.3 Measurement

Now we can define mathematically what a measurement is. It is simply the map

$$\mathcal{A} \times \mathcal{S} \to \mathcal{P}(\mathbb{R}); \quad (A, M) \mapsto \mu_A; \mu_A(E) := \operatorname{tr}(MP_A(E)).$$

What is **not** simple, is giving a **physical or philosophical interpreting** of what it means to measure. This is where one can get into a discussion of collapsing wave functions or multiple worlds.

Also, one should check that μ_A is a measure because the projection measures are measures. But the reason why $0 \leq \mu_A(E) \leq 1$ is because of the $\operatorname{tr} A = 1$ condition.

As in the classical setting, M gives a **pure state** if M is projection onto some unit vector ψ. **Mixed states** are just convex linear combinations of pure states. I think that some physicists may call H the space of states or pure states for this reason.

3.4 Commutativity and Simultaneous Measurement

Let's return to the issue of commutativity. If we have two unbounded self-adjoint operators A, B, what should it mean for them to commute? We would have to make sure their domains matched up somehow via restriction but that would be a pain. What we'll mean then is that A, B commute if $P_A(E)$ and $P_B(E')$ commute for all E, E'.

It turns out this second condition holds if and only if $\exp(iuA)$ and $\exp(ivB)$ commute for all $u, v \in R$. Note, by the way that each of these is a unitary operator because self adjoint operators have real eigenvalues; i.e. U is unitary if $UU^* = U^*U = \mathrm{id}$. Now, in finite dimensions, $UU^* = \mathrm{id} \Rightarrow U^*U = \mathrm{id}$. But not so in infinite dimensions.

Why do we care about commutativity? The reason is that we may have several observables $A_1, ..., A_n$ and we want to assign a probability measure to all of them **simultaneously**. i.e. we want to measure all of these observables at once. It turns out, we can define the measure

$$\mu_{A_1,...,A_n}(E_1 \times ... \times E_n) = \operatorname{tr}(MP_{A_1}(E_1)...P_{A_n}(E_n)).$$

But this won't be a true measure unless all the projections commute. Thus, we need commutativity for simultaneous measurement. This is reminiscent of the fact that we can simultaneously diagonalize if we have commuting operators.

3.5 Dynamics

At last we turn to the dynamics of quantum mechanics. We'll find that on a formal level, there is no difference from classical mechanics. We'll see that the equations look nearly identical to the classical mechanics equations! We have two formulations which also depend on a Hamiltonian. Assume $H \in \mathcal{A}_0$ (bounded operators in $\mathcal{A}$) is given; it will be independent of time so that we may assume that energy is conserved and thus, we're in a closed system. The equations are:

1. Heisenberg formulation:
$$\frac{dM}{dt} = 0, \quad \frac{dA}{dt} = \{H, A\}_\hbar$$
2. Schrödinger formulation:
$$\frac{dA}{dt} = 0, \quad \frac{dM}{dt} = -\{H, M\}_\hbar$$

We might need A, M to be bounded operators as well, at least, if we want the equations to be easier to study. But the main point of interest is that these equations are **linear** whereas in classical mechanics, they are nonlinear ODES. This is of course because the bracket is bilinear and the operators are linear.

However, do not be deceived! Despite the linearity of the equations, they are not easy to fully understand. To see why, let us study this more closely. As usual, the dynamics should be given by some 1-parameter group of diffeomorphisms; they should tell us how the system evolves over time. Let's consider the Heisenberg setting and let $U(t) = \exp(-\frac{i}{\hbar}tH)$; it is a 1-parameter group of unitary operators. Let $A_0 = A|_{t=0}$.

Let's take a guess at a solution. Let $A(t) = U(t)^{-1}A_0U(t)$.

Accepting this, then $U'(t) = -\frac{i}{\hbar}HU$. Note that since U is some exponential of H, then H commutes with U and also U^{-1}. Therefore, $A'(t) = \frac{i}{\hbar}(HA - AH) = \{H, A\}_\hbar$. Thus, one sees that $A = U^{-1}A_0U$ is a solution and in some sense, we've "solved" the equations.

But it's no good if we don't understand the solution and the difficulty lies in understanding U. To understand it, we basically need a full understanding of its spectrum. Questions to ask include whether the spectrum is discrete, bounded, etc.

The Schrödinger formulation is quite similar and we'll have $M(t) = U(t)M_0U(t)^{-1}$. In any event, these two formulations are equivalent which is quite a profound claim. It means that on one side, we can fix our states and let the observables vary or we can fix our observables and let the states vary. Mathematically, we're saying: $\mu_{A(t)}(E) = \mu(t)_A(E)$. The left hand side is equal to $\mathrm{tr}(M_0P_{A(t)}(E))$ while the right hand side is equal to

$$\begin{aligned}\mathrm{tr}(M_0U(t)^{-1}P_{A_0}(E)U(t)) &= \mathrm{tr}(U(t)M_0U(t)^{-1}P_{A_0}(E)) \\ &= \mathrm{tr}(M(t)P_{A_0}(E))\end{aligned}$$

The first equality here comes from the cyclic property of trace.
Claim: When M_0 is a pure state; i.e. $M_0 = P_{\psi_0}$ where $\psi_0 \in \mathcal{H}$, then $M(t) = P_{\psi(t)}$ where $\psi(t) = U(t)\psi_0$. In other words, pure states just get "rotated around."

Proof. It's clear that $M(t)\psi(t) = U(t)M_0U(t)^{-1}U(t)\psi_0 = U(t)M_0\psi_0 = U(t)\psi_0 = \psi(t)$. The penultimate equality comes by the fact that projecting ψ_0 onto $\langle\psi_0\rangle$ does nothing. Now let $\phi \perp \psi(t)$. This means that $U(t)^{-1}\phi \perp \psi_0$ because unitary operators preserve the inner product. This means that $M(t)\phi = UM_0U^{-1}\phi = 0$ since projecting an orthogonal vector onto $\langle\psi_0\rangle$ does nothing. Thus, $M(t) = P_{\psi(t)}$. □

Important Note: This discussion is for **closed** systems. We have a time-dependent unitary operator which gives us a forward time-evolution and we could also run it backwards. But if this were an open system which interacts with the environment, we should not have a way to invert the time evolution. It is like dropping a glass of water onto the ground. We do not expect it possible for the glass to reform, for the water to be unspilled, and for the whole thing to travel back to ones hand.

4 Schrödinger's Equations

The time evolution of pure states is determined by the time dependent Schrödinger equation. Letting $\psi = \psi(t, x)$, the equation is:

$$i\hbar \frac{d\psi}{dt} = H\psi.$$

Note that if $\psi(t) = U(t)\psi_0$, then $d\psi/dt = -\frac{i}{\hbar}HU(t)\psi_0 = -\frac{i}{\hbar}H\psi$. So a solution to the time dependent Schrödinger equation comes from unitary evolution of a vector and the $\psi(t)$ gives us pure states $M(t) = P_{\psi(t)}$.

On the other hand, if we have a stationary pure state, $M(t) = M_0 = P_{\psi_0}$, then we'll find that after normalizing ψ_0, we can let $\psi(t) = U(t)\psi_0 = c(t)\psi_0$ where $|c(t)| = 1$. Therefore, let $c(t) = \exp(-\frac{i}{\hbar}\lambda t)$ where $\lambda \in \mathbb{R}$. Then, ψ_0 determines stationary states if and only if $U(t)\psi_0 = \exp(-\frac{i}{\hbar}\lambda t)\psi_0$. If we apply the operator $i\hbar\frac{d}{dx}$ to this last equation, we'll get $\lambda\psi_0$. Thus, these stationary pure states are determined by ψ_0 which are solutions to the time independent stationary Schrödinger equation: $H\psi_0 = \lambda\psi_0$.

Indeed, the λ are eigenvalues and the ψ_0 are eigenfunctions. These stationary states are sometimes called bound states in physics.

One comment to make: the Schrödinger equation is a PDE; it has t and x variables. One might ask; can we separate t and x to solve the equation? This is a common technique used in PDE theory. The answer is essentially, yes, in quantum mechanics. But in quantum field theory, when there are Lorenzian transformations that exchange space and time coordinates, we are not able to do this.

One other comment to make: the Schrödinger equation can be derived from the following assumptions:

1. $m\frac{d}{dt}\langle x\rangle = \langle p\rangle$
2. $\frac{d}{dt}\langle p\rangle = -\langle V'(x)\rangle$
3. $[x, p] = i\hbar$

Here, $\langle x\rangle, \langle p\rangle$ mean the expectation value for position and momentum. So the first equation looks like the classical Newtonian equation but with expectation values (quantum mechanics should approximately return classical results). The second equation is similar since force is given by $F = -V'(x)$ where V is a scalar potential. These two equations are called the Ehrenfest equations. The last equation is just to say that x and p do not commute. On the other hand, assuming noncommutativity and the Schrödinger equations, one can derive the Ehrenfest equations.

5 Heisenberg Uncertainty Relations

We come now to one of the foundational principles in quantum mechanics. The Heisenberg Uncertainty Principle is often stated loosely as saying: "The more accurately we know the

position of a particle, the less accurate we know the momentum of the particle. And vice verse." We'll see now that this principle is actually completely mathematical and not just a feature of the physical world.

Let A, M be our observable and state. The **expected value** of the two is defined as:

$$\langle A|M\rangle = \int_{-\infty}^{\infty} \lambda d\mu_A(\lambda).$$

It is a real value but can also be ∞. It turns out that if $M = P_\psi$ is a pure state, then $\langle A|M\rangle = (A\psi, \psi)$ if $\psi \in D(A)$. Otherwise, we let it be ∞.

The **variance** of an observable is defined as:

$$\sigma_M^2(A) = \langle (A - \langle A|M\rangle \,\mathrm{id})^2 | M\rangle.$$

Of course, if $\langle A|M\rangle$ is infinite, we can't do this but otherwise, we do. Also, this is always a nonnegative value.

Theorem 5.1. *If $M = P_\psi$ is a pure state and ψ is in all the relevant domains concerning self-adjoint operators A, B, then*

$$\frac{\hbar^2}{4}\langle \{A, B\}_\hbar | M\rangle^2 \leq \sigma_M^2(A)\sigma_M^2(B).$$

Proof. This proof is due to Hermann Weyl. We may assume that $\langle A|M\rangle = \langle B|M\rangle = 0$ (I think because ψ is just a single vector). Now for any $\alpha \in \mathbb{R}$, we consider

$$0 \leq \|(A + i\alpha B)\psi\|^2.$$

Expanding the right hand side, we have:

$$\begin{aligned}(A\psi, A\psi) + \alpha^2(B\psi, B\psi) + i\alpha(B\psi, A\psi) - i\alpha(A\psi, B\psi) =& (A^2\psi, \psi) + \alpha^2(B^2\psi, \psi) \\ &+ \alpha(i(AB - BA)\psi, \psi) \\ = \sigma_M^2(A) + \alpha^2\sigma_M^2(B) &+ \alpha(\hbar\{A, B\}_\hbar\psi, \psi)\end{aligned}$$

Thus, we see that we have the following inequality:

$$0 \leq \sigma_M^2(B)\alpha^2 + \langle \hbar\{A, B\}_\hbar | M\rangle\alpha + \sigma_M^2(A).$$

This implies that the discriminant of the quadratic function in α must be nonpositive. Thus,

$$\hbar^2\langle \{A, B\}_\hbar | M\rangle^2 - 4\sigma_M^2(A)\sigma_M^2(B) \leq 0$$

or

$$\frac{\hbar^2}{4}\langle \{A, B\}_\hbar | M\rangle^2 \leq \sigma_M^2(A)\sigma_M^2(B).$$

□

The main observation is: if A, B don't commute, then the left hand side is positive and so there must be some variance for both A and B. Moreover, if the variance of one is relatively small, the variance of the other must be relatively large.

6 Quantization

6.1 Preliminary Points

We begin in a classical mechanical system and we wish to make it quantum mechanical in some sense. Before we discuss a more generalized theory, there are a few remarks to make. Suppose we begin with a symplectic vector space (V, ω). To quantize this, we want to assign a complex Hilbert space to (V, ω) and we wish this assignment to be as functorial as possible. Moreover, if two wavefunctions differ by a constant phase, they give the same observable and hence, we should quotient out the $U(1)$-action; the operator should reflect this. This amounts to asking for a **projective** Hilbert space.

But what exactly does this functoriality wish mean?

1. We might request that the symmetries of the symplectic vector space V be represented as unitary operators on the Hilbert space. So this means we want to find a projective representation of the symplectic group. One way of constructing this is to take a splitting of $V \cong L \oplus L'$, the sum of two Lagrangian subspaces. With respect to this splitting (called a **polarization**), we may let the Hilbert space be L^2 functions on L. It turns out that the underlying projective space does not depend on the splitting. This statement is equivalent to the existence of a projective unitary representation of the symplectic group, called the **metaplectic** representation. The quantum Hilbert space is the underlying complex Hilbert space of that representation.

2. Another interpretation of the functoriality request is to use a dual picture. Instead of studying points of V and points in the corresponding Hilbert space $\mathcal{H}$, we can study functions and operators, respectively. Functions on a symplectic vector space form a Poisson algebra. In this view, quantization should assign $f \in C^\infty(V, \mathbb{R})$ to a self-adjoint operator $\hat{f}$ on $\mathcal{H}$.

   Naively, we might ask that the Poisson bracket is mapped to the commutator bracket. But experience shows that it is best to introduce a parameter $\hbar$ and work over formal power series in $\hbar$. Then, we relax the request and only ask that the Poisson bracket is mapped to the commutator bracket up to first order in $\hbar$.

With this preamble, let's dive in. We have a phase space M with a Poisson bracket $\{,\}$ and algebra of classical observables $A^0 = C_c^\infty(M)$; here, we're taking compactly supported functions. Hence, we have a triple $(M, \{,\}, A)$.

6.2 Geometric Quantization and Boothby-Wang Bundles

Here, let's follow the dual picture for awhile and try a quantization with geometric flavor. If (M, ω) is a symplectic manifold and $f : M \to \mathbb{C}$ is a function (which determines the dynamics), then let $\hat{f}$ be an operator on $L^2(M, \mathbb{C})$. We let X_f be the ω-dual of f. Then $\hat{f}(\psi) := X_f \cdot \psi = d\psi(X_f)$. Then, if $h := \{f, g\}$, $\hat{h}(\psi) = [X_f, X_g] \cdot \psi$. On the other hand, this is what $[\hat{f}, \hat{g}]$ does to ψ. So $\widehat{\{f, g\}} = [\hat{f}, \hat{g}]$.

But note that this sends the constant function $\mathbb{1}$ to the zero operator. But the constant function $\mathbb{1}$ is the identity element in the algebra and we would like to send it to the identity operator. So let us try introducing an extra dimension. We'll let $\pi : Y \to M$ be an S^1-bundle with a connection 1-form λ that has curvature $\pi^*\omega$ (hidden here is that we need ω to be an integral cohomology class; this is basically equivalent to the existence of such a bundle $Y \to M$). Then X_f, an infinitesimal symplectomorphism, lifts to Y by an infinitesimal

quantomorphism; this is a vector field that preserves λ. If (Y, λ) is a contact manifold, then the Reeb vector field should do the trick.

Anyways, we lift X_f to obtain $\xi_f = X_f \oplus f\xi$ where ξ is an infinitesimal generator of the action of S^1 on Y. If we consider a subspace of S^1-equivariant maps $\psi : Y \to \mathbb{C}$ and define $\hat{f}(\psi) = \xi_f \cdot \psi$, then $\widehat{\{f, g\}} = [\hat{f}, \hat{g}]$ still holds. Moreover, $\xi_{\mathbb{1}} = \xi$ and $\hat{\mathbb{1}}(\psi) = \xi \cdot \psi = \psi$ because of the S^1-equivariance.

However, we are not finished with quantization yet because ψ should really only depend on the configuration variables that make up half the dimension of M. So the bundle $Y \to M$ which mathematicians call a Boothby-Wang bundle, is known to physicists as a **prequantization** bundle. This difficulty with the dimensions seems to be a rather major one in geometric quantization. One attempt at resolving this is to find a Lagrangian foliation $M \to Q$ and consider functions that only depend on Q. Such functions are called **polarized**. This cuts our dimension by half and in the classical case of a phase space, the map is simply $(q, p) \mapsto q$.

If (V, ω) is simply a symplectic vector space, the prequantized bundle is trivial: $V \times S^1$ and the connection is $\lambda = \alpha + \frac{1}{iz}dz$ where $\omega = d\alpha$ and $z = e^{i\theta}$ and hence, $dz = iz d\theta \Leftrightarrow d\theta = \frac{1}{iz}dz$. The simplest polarizations to take are linear Lagrangian subspaces.

This simple situation is nice but if we want to represent some 1-parameter family of symplectomorphisms, we want the polarization to be invariant. But this hardly ever happens. For example, the harmonic oscillator has Hamiltonian which rotates the Lagrangian subspace (much like how we can rotate the polarization plane of light). For the harmonic oscillator, there is a fix but in general, geometric quantization has not been successfully implemented in all cases.

6.3 Formal Quantizaiton

Quantization of this triple is a **map** $Q_\hbar : A^0 \to \mathcal{A}_\hbar^0$ which goes from classical observables with compact support to quantum observables. This $\mathcal{A}_\hbar^0$ will be **bounded** self-adjoint operators on some Hilbert space $\mathcal{H}$.

Before getting into the properties we require of $Q_\hbar$, we make some remarks. First, A_0 is a Lie algebra with respect to $\{,\}$ while $\mathcal{A}_0$ is a Lie algebra with respect to $\{,\}_\hbar := \frac{i}{\hbar}[,]$ (the commutator). It is important to note at this point that whatever properties $Q_\hbar$ has, it cannot be a Lie algebra morphism because if it were, then this would basically say that the classical mechanics embeds into quantum mechanics (after some quotienting if needed); this is point (2) addressed at the start of this section. Also, $\hbar > 0$ is usually a fixed value for physicists but for mathematicians, this can be a parameter and thus, we get a family of maps $Q_\hbar$.

We now state the properties we require of $Q_\hbar$:

1. $Q_\hbar$ is injective. On a physical level, we don't want two different classical observables to map to the same quantum observable.

2. Let $A, B \in \mathcal{A}_\hbar^0$. We may define the **Jordan product:** $A \circ B := \frac{1}{2}AB + BA$ (here, we're using operator product); this is commutative but not necessarily associative. Given $f, g \in A^0$, we require that
$$Q_\hbar^{-1}\Big(Q_\hbar(f) \circ Q_\hbar(g)\Big) = fg.$$

3. For all $f, g \in A^0$, we require that
$$\lim_{\hbar \to 0} Q_\hbar^{-1}\Big(\{Q_\hbar(f) \circ Q_\hbar(g)\}_\hbar\Big) = \{f, g\}.$$

This is known as the Correspondence Principle which Niels Bohr probably stated.

These properties seem somewhat mysterious though one can see that they require $Q_\hbar$ to preserve product and bracket structures in some sense. We can also define a new multiplication on $C^\infty(M)$ using $Q_\hbar$. Let $f, g \in C_(^\infty M)$. Then

$$f *_\hbar g := Q_\hbar^{-1}\Big(Q_\hbar(f)Q_\hbar(g)\Big) = fg + \frac{\hbar}{2}\{f,g\} + \{\text{higher order terms}\}.$$

Thus, we see that we can define a new multiplication $*_\hbar$ which is a deformation of usual multiplication of functions. This deformation even includes some data of the original Poisson bracket.

Let's talk more abstractly for a moment. Let A be a commutative algebra and A_t $(t \geq 0)$ be a family of algebras which are isomorphic as vector spaces to $A_t \cong A \otimes \mathbb{C}[\![t]\!]$ but with a different multiplication: $a *_t b = ab + tm_1(a,b) + t^2 m_2(a,b) + t^3 m_3(a,b) + ...$ We would like $*_t$ to be associative. In such a setting, we call this family a **deformation quantization**. Note that there is no Hilbert space involved, unlike the quantization we had from earlier. Therefore, this deformation quantization is not properly "physical quantization" but is rather, a formal quantization.

Regardless, the associativity actually implies that m_1 is a cocycle (in some Hochschild sense...). Now, we can ask ourselves a question: **"Given an algebra of classical observables and m_1, a Poisson bracket, is it always possible to find a deformation quantization where the multiplication is associative?"**. Framed a different way, we're given m_1 and we'll like to find $m_2, m_3, ...$ such that the multiplication is associative. Note that we're not doing this with reference to any Hilbert space. So we're in the abstract setting.

It was long known that if our algebra $A = C^\infty(M)$ comes from a symplectic manifold (M, ω), it is possible. However, it wasn't until 1997 when Maxim Kontsevich showed that this is possible when $(M, \{,\})$ is a Poisson manifold. Not only did he show it was possible, he gave a formula for the product which involves studying graphs and differential forms on configuration spaces of finitely many points.

7 "Philosophy"

In this last section, we discuss a little bit of the "philosophy" of how we approach these physical theories. I don't mean philosophy in the academic sense. Quantum mechanics is fundamental while classical mechanics can be seen as a limit of quantum mechanics in some sense ($\hbar \to \infty$). We might say that many quantum systems are simply "quantizations" of classical systems. This means that we do some perturbation to make the algebra noncommutative. But there are examples of quantum mechanical systems which do not arise via quantization of classical systems. I'm not sure if Takhtajan meant this in a physical sense or mathematical. In any event, this suggests that those quantum systems are more fundamental than the classical systems. Moreover, quantization is much more of a mathematical notion than a physical notion at the end of the day in which we attempt to move from a mathematically classical setting to the quantum setting.