# Examples from Topology

Sam Auyeung

February 5, 2020

## 1 Construction of $S^{n+m}$ from $S^n \times S^m$

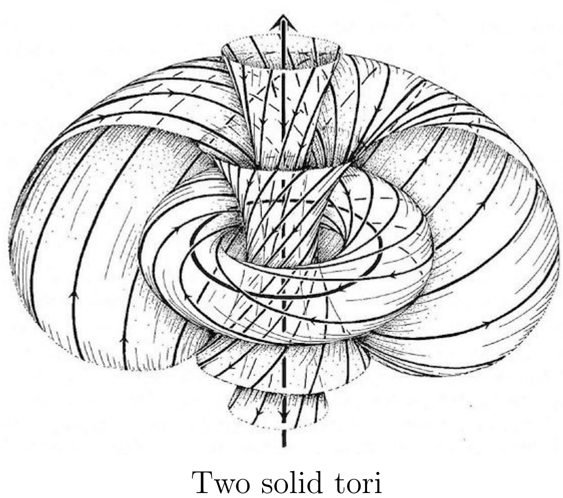
Consider the homology of $S^n \times S^m$: by the Künneth formula, we get homology in $0, n, m, n+m$, all of rank one. That is, $S^n \times S^m$ has cell decomposition $e_0 \cup e_n \cup e_m \cup e_{n+m}$. On the other hand, $S^n \vee S^m$ is a wedge and so its cell decomposition is $e_0 \cup e_n \cup e_m$. Then when we take the quotient $S^n \times S^m / S^n \vee S^m$, we get $e_0 \cup e_{n+m}$. But the boundary of the $e_{n+m}$ was in the $S^n \vee S^m$ so the boundary is crushed to a point and we get $S^{n+m}$. Thus, $S^n \times S^m / S^n \vee S^m$ is homeomorphic to $S^{n+m}$. This type of construction is called a **smash product.**

## 2 Heegard Decomposition of $S^3$ and Zero Surgery

### 2.1 Heegard Decomposition

We can decompose $S^3$ in many ways. The genus-0 way is to decompose it into two balls. Consider $U = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ and $V = \{(x, y, z) \in \mathbb{R}^3 : z \leq 0\}$; if we glue the $xy$-plane of $U$ and $V$ together by the identity map, we'll get all of $\mathbb{R}^3$. $U$ and $V$ without their boundaries are both homeomorphic to the open ball $B^3$. By adding a point at infinity in $\mathbb{R}^3$, the $xy$-plane wraps into $S^2$.

The genus 1 decomposition is more interesting. Consider $S^3$ as $\mathbb{R}^3$ with a point at infinity. If we remove a solid torus $S^1 \times D^2$ from $\mathbb{R}^3$ (centered on the $z$-axis, say), what remains has the torus $T^2$ as boundary. Well, what has $T^2$ as boundary? Another solid torus. It's more difficult to imagine this solid torus so consider the following picture (from Okinama Institute of Science and Technology website):



Two solid tori

Let $A$ be the removed solid torus and $B$ the remaining solid torus. We can fill the hole of $A$ with some line segments parallel to the $z$ axis. This gives us a trivial bundle of lines over a disk $D^2$ which lies in the $xy$-plane. We can then extend and curve the line segments so that the two ends of a given line segments join together into an $S^1$. All these $S^1$'s are linked around $A$ and to ensure they are disjoint, some of the $S^1$'s have to have large radius. In fact, for the fibers closer to $0 \in D^2$, the radius grows. In particular, the $z$-axis goes through 0 and forms a circle with infinite radius (it compactifies on a point at $\infty$). So now we have our other solid torus: $S^1 \times D^2$.

## 2.2 Zero Surgery

Suppose we started out with two solid tori $S^1 \times D^2 \cup S^1 \times D^2$. Let's call the first circle $\alpha$ and the second one (boundary of $D^2$) $\beta$. Now, we want to glue the two solid tori along their boundaries. We could, for example, glue by interchanging $\alpha$ and $\beta$: $(\alpha, \beta) \mapsto (\beta, \alpha)$. This would give us $S^3$.

On the other hand, if we glue using the map $(\alpha, \beta) \mapsto (\alpha, \beta)$, the identity map, we get $S^1 \times S^2$! The way to see this is, consider our set up from the decomposition. The solid torus $B$ has $D^2$ in the $xy$-plane and the $S^1$'s are linked around $A$. Let's remove the interior of $D^2$ and fill it into one of the $S^1$'s linked around $A$. This particular $S^1$ corresponds to an $S^1$ curve on $A$ which also bounds a disk. Now glue the newly filled in disk to the corresponding disk in $A$ along the boundary. This is simply gluing two disks along their boundary which gives $S^2$. Now if we do this for all the links, we get $S^1 \times S^2$.

This process can also be thought of as starting with an unknot in $S^3$ and thickening it to obtain a tubular neighborhood. Remove this neighborhood which gives $S^1 \times D^2$. Now glue back a solid torus but now as $D^2 \times S^1$ (we fill in the other $S^1$ instead).

I imagine this type of surgery can be generalized to using different sorts of knots.

# 3 The Hopf Fibration

**Example 3.1.** Recall the Hopf Degree Theorem:

**Theorem 3.2.** *Let $M$ be a compact, connected, orientable $n$-manifold. Let $f, g : M \to S^n$ be continuous maps. Then $f \simeq g$ (homotopic) if and only if* $\deg f = \deg g$.

The degree of a map can be defined in many ways between any compact, connected, orientable $n$-manifolds. For example, if $f : M \to N$ is a smooth map, then given a regular value $y$,

$$\deg f = \sum_{x \in f^{-1}(y)} \text{sign } df_x.$$

Or take a top form $\eta$ of $N$. Then

$$\int_M f^* \eta = k \int_N \eta$$

for some $k \in \mathbb{Z}$. Define $\deg f := k$. But in fact, degree can be defined for continuous maps between topological manifolds. One notes from the second definition above that the degree of a map should depend on just the top de Rham cohomology. Thus, we could define degree instead by considering $f_* : H^n(M; \mathbb{Z}) \to H_n(N; \mathbb{Z})$ (both top homologies are $\mathbb{Z}$). The fundamental class $[M]$ is mapped to some multiple of the fundamental class of $N$: $k[N]$. Again, let $\deg f := k$. The point is that the degree of a map only considers top cells and the topological information of lower cells is lost. However, $S^n$ only has 0 and top homology. Thus, in this setting, the degree of a map is a powerful homotopy invariant for maps.

One may wonder what degree of a map can tell us about maps $f, g : S^n \to M$. Is there an inverse Hopf Degree Theorem? No, it fails. Here is a counterexample. In general, if $M$

is a manifold, then the map $\tau : M \times M \to M \times M$ which sends $(x, y) \mapsto (y, x)$ is a deg $-1$ "twisting" map.

Now, define $g : S^2 \times S^2 \to S^2 \times S^2$, $g(x, y) = (y, h(x))$ where $h : S^2 \to S^2$ is of degree $-1$. This $h$ is easily constructed: take the map $S^1 \to S^1$ sending $z \mapsto 1/z$. It is deg $-1$. Then take the suspension of the map to obtain $h : S^2 \to S^2$. So $\deg g = (\deg h)(\deg \tau) = +1$ but $g$ is not homotopic to identity.

Now, $\pi_4(S^2) = \mathbb{Z}_2$. Then, $\pi_4(S^2 \times S^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $f : S^4 \to S^2 \times S^2$ be a map which is represented by $(1, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $g \circ f$ represents $(0, 1)$. $\tau$ swaps the two generators $(1, 0)$ and $(0, 1)$ and I think $h$ sends $(0, 1) \mapsto (0, -1) = (0, 1)$. The maps are not homotopic but have the same degree (as $g$ is degree one).

This construction could be generalized to $S^{2n} \to S^n \times S^n$ because it seems $\pi_{2n}(S^n) \neq 0$ for at least some $2 \leq n \leq 7$, but possibly for all. However, it seems that in the construction above, we needed that $(0, -1) = (0, 1)$ in the group.

**Example 3.3.** What is $\pi_3(\bigvee_{i=1}^n S_i^2)$? When $n = 1$, we have just one 2-sphere and the Hopf fibration tells us this group is $\mathbb{Z}$. Let $X := \bigvee^n S^2$.

Consider a map $f : S^3 \to X$. It is homotopic to a smooth map. The Thom-Pontryagin construction tells us to take choose a point on each $S^2$ (not the wedge point); we can perturb $f$ such that it is transverse to each of these points $x_i$ and in fact, transverse in a neighborhood. This means that $L_i = f^{-1}(x_i)$ is a 1-manifold. It may be disconnected but they should be disjoint unions of $S^1$. We call these links.

Given two links, we can consider their linking number. $S^3$ is simply connected so all the components of the links contract. Thus, given links $L_1$ and $L_2$, extend $L_1$ to a disk and orient $L_2$. Then, $L_2$ intersects the disk of $L_1$ some number of times; we assign $\pm 1$ to the intersections depending on orientation and we get a number out called the linking number: $lk(L_1, L_2)$. Claim: $lk(L_1, L_2) = lk(L_2, L_1)$. We can also define self-linking number: choose a normal vector field to a link $L$ (or just a frame: non-tangent vectors to $L$). Let $L'$ be $L$ pushed along the normal vector field. We define $lk(L, L) := lk(L, L')$.

We may then form a symmetric matrix of these linking numbers. The claim is that these maps $f$ are in 1-1 correspondence with symmetric $n \times n$ matrices with $\mathbb{Z}$ coefficients up to change of basis; i.e. $A^t BA$. But we don't want to just consider $f$ but its homotopy class. If we have a homotopy between $f$ and $f_1$, what does the picture look like in $S^3$? That is, how do the links move about and what sort of surface do they sweep out? The answer is that the links of $f_1$ are cobordant to links of $f$.

**Proposition 3.4.** *Let $G$ be the group under addition of $n \times n$ symmetric matrices with $\mathbb{Z}$ coefficients, modulo $GL(n, \mathbb{Z})$. $\pi_3(\bigvee^n S^2) \cong G$. In particular, when $n = 1$, $\pi_3(S^2) = \mathbb{Z}$.*

# 4 Whitehead's Theorem

**Example 4.1.** Recall Whitehead's theorem: Let $X, Y$ be CW complexes. If there is a continuous map $f : X \to Y$ which induces isomorphisms on every homotopy group; i.e. $f_* : \pi_k(X) \cong \pi_k(Y)$ for all $k$, then $X$ and $Y$ are homotopy equivalent.

Is it possible then, to have two spaces which have all the same homotopy groups but are not homotopy equivalent? Yes. We consider $S^2$ and $X = \mathbb{C}P^\infty \times S^3$. These are CW complexes. The Hopf fibration and long exact sequence of homotopy groups for fibrations give us that $\pi_k(S^2) \cong \pi_k(S^3)$ for $k \geq 3$. Since $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, then the only homotopy group it has is $\pi_2 = \mathbb{Z}$. Thus, for $k \geq 3$, $\pi_k(X) \cong \pi_k(S^2)$. And of course $\mathbb{Z} = \pi_2(S^2) \cong \pi_2(X)$ and $\pi_1(S^2) = \pi_1(X) = 0$ (same for $\pi_0$). Thus, they have all the same homotopy groups.

However, the cohomology ring of $\mathbb{C}P^\infty$ is just the polynomial ring $\mathbb{Z}[x]$ and so $H^*(X)$ is nonzero for all $*$, or at least, infinitely many. But $H^*(S^2)$ is zero for $* > 2$. Thus, they have different homology and thus, are **not** homotopy equivalent despite having all the same homotopy groups. So there is no single map which induces the isomorphisms on all homotopy groups, by the contrapositive of Whitehead's theorem.

By the way, the generator of $H^*(\mathbb{C}P^\infty, \mathbb{Z})$ has an interesting Poincaré dual: the direct limit of hyperplanes $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$.

We have an even simpler example which involves finite CW complexes; in fact, manifolds. Let $X = S^2 \times \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \times S^3$. Because $S^n \to \mathbb{R}P^n$ is a covering, $\pi_k S^n \cong \pi_k \mathbb{R}P^n$ for $k \geq 2$. Moreover, the product formula for homotopy groups is more straightforward than for (co)homology. Thus, $\pi_k X \cong \pi_k Y$ for all $k$; in particular, when $k = 1$, the groups are isomorphic to $\mathbb{Z}_2$.

On the other hand, $X$ is orientable while $Y$ is not and so they have different $H^5_{dR}$ groups and hence, are not homotopy equivalent.

# 5 Thom Isomorphism

This section is mostly from Bott-Tu. Let $\pi : E \to M$ be an oriented vector bundle of rank $n$. We can define something called compact vertical cohomology on the total space $E$. The elements of $\Omega^*_{cv}(E)$ will be forms $\alpha$ which satisfy the following: for every compact $K \subset M$, $\pi^{-1}(K) \cap \mathrm{Supp}(\alpha)$ is compact. In particular, the forms have compact support when restricted to each fiber. Note that this does not mean that the forms have compact support. Consider a form on the total space of $pr_1 : \mathbb{R}^2 \to \mathbb{R}$ which has support between the curves $y = \pm x$.

Now, suppose that $E = M \times \mathbb{R}^n$ with coordinates $(t_1, ..., t_n)$ on $\mathbb{R}^n$. We define a map $\pi_*$ which is essentially integration along the fibers. The forms in $\Omega^*_{cv}(E)$ fall into two types.

1. $(\pi^*\phi) f(x,t) dt_{i_1} \wedge ... \wedge dt_{i_r}$ where $r < n$. Here, $\phi$ is a form on $M$ and $f$ is a function with compact support for each fixed $x \in M$. Note that at least all forms of deg $< n$ are of this type.

2. $(\pi^*\phi) f(x,t) dt_1 \wedge ... \wedge dt_n$; $\phi$ and $f$ are as above. Note that top forms are of this type.

We define $\pi_*$ to map Type 1 forms to 0 and Type 2 forms to $\phi \int_{\mathbb{R}^n} f(x,t)\, dt_1 ... dt_n$. It's clear that by defining $\pi_*$ on patches $U \times \mathbb{R}^n$, we can define $\pi_*$ on general orientable vector bundles.

It turns out that $\pi_*$ commutes with $d$ and so it is a chain map. Moreover, we have a proposition:

**Proposition 5.1.** *Let $\pi : E \to M$ be an oriented rank $n$ vector bundle, $\tau \in \Omega^*(M)$, $\omega \in \Omega^*_{cv}(E)$.*

1. *(Projection Formula): Then $\pi_*((\pi^*\tau) \wedge \omega) = \tau \wedge \pi_*\omega$.*

2. *If $M$ is also oriented of dimension $m$, $\omega \in \Omega^q_{cv}(E)$, and $\tau \in \Omega^{m+n-q}_c(M)$, then using the orientation on $E$ induced by $M$ and also the orientation of the bundle, we have*

$$\int_E (\pi^*\tau) \wedge \omega = \int_M \tau \wedge \pi_*\omega.$$

**Theorem 5.2** (Thom Isomorphism)**.** *If $\pi : E \to M$ is an orientable rank $n$ vector bundle and $M$ is of finite type, then*

$$H^*_{cv}(E) \cong H^{*-n}(M).$$

The Thom isomorphism $T$ is the inverse of $\pi_*$. We define the **Thom class** to be $\Phi := T(1)$ where $1 \in H^0(M)$ is the constant function 1. Because of the projection formula, we have that $\pi_*((\pi^*\tau) \wedge \Phi) = \tau \wedge \pi_*\Phi = \tau$. This means that the Thom isomorphism can be written as $T(\omega) = \pi^*\omega \wedge \Phi$. Thus, if we have a description of the Thom class, we can define $T$.

**Proposition 5.3.** *The Thom class $\Phi \in H^n_{cv}(E)$ for a rank $n$ oriented vector bundle $\pi : E \to M$ is uniquely characterized as the cohomology class which restricts to the generator of $H^n_c(F)$ on each fiber $F$.*

The Thom class restricted to a fiber is something like $\rho(t)dt_1 \wedge ... \wedge dt_n$ where $\rho$ is a bump function and the integral of this form is 1. As a visualization, one might imagine $\Phi$ as being a bump form in a neighborhood of the zero section.

There are many reasons why the Thom class is important. To begin, we can realize the Poincaré dual of a submanifold as the Thom class of the normal bundle. Recall that if $S$ is a $k$-submanifold in $M^m$, then it's Poincaré dual is a $(m-k)$-form $\eta_S$ which satisfies

$$\int_S \omega = \int_M \omega \wedge \eta_S$$

for every $k$-form $\omega$. The claim is that the Poincaré dual of $S$ is represented by the Thom class of the normal bundle of $S$.

Another way the Thom class can appear is as the Euler class of a oriented rank 2 vector bundle. Let $E \to M$ be such a bundle where $M$ is also oriented. Then pulling the Thom class back to $M$ via the zero section gives the Euler class of $E$. More generally, if we have the same situation but the rank is $n$, then the Euler class of $E$, $e(E) \in H^n(M)$, is Poincaré dual to the zero locus of a transverse section. The proof uses the Thom class.

# 6 Cobordism

**Example 6.1.** Any manifold $M^n$, if it's the boundary of a $(n+1)$-manifold, is cobordant to $S^n$; just cut out a small ball in the larger manifold and you've created an $S^n$. This automatically tells us that if $M, N$ are $n$-manifolds which bound some manifolds, we can glue the manifolds they bound along some $S^n$ and now $M$ and $N$ are cobordant to each other.

Do we have some descriptions for when a manifold is the boundary of another manifold? Yes. By Pontryagin and Thom's work: All the Stiefel–Whitney numbers of a smooth compact manifold $X$ vanish if and only if the manifold is the boundary of some smooth compact (possibly unoriented) manifold. Note that all the Stiefel-Whitney classes don't have to vanish; just the SW numbers.

In the case of closed, oriented 3-manifolds, there's an amazing result that they are all parallelizable. That precisely means the tangent bundle is trivial (or they admit global frames). So any closed, oriented 3-manifolds is the boundary of something and all the of these 3-manifolds are cobordant to each other. Put another way, the oriented cobordism group of 3-manifolds $\Omega_3^{SO} = 0$ (here, SO means the special orthogonal which involves orientation).

Even more amazing, Thom computed the unoriented cobordism ring of manifolds $\Omega_*^O$. Everything has 2-torsion since $2M$ is the boundary of $M \times I$. He showed, with some pretty amazing arguments involving Thom spaces, Eilenberg-MacLane spaces, Postnikov towers, and stable homotopy groups, that $\Omega_*^O = \mathbb{Z}_2[\hat{X}_1, X_2, \hat{X}_3, X_4, ..., \hat{X}_{2^n-1}, ...]$. So, this is a ring in which we take out all generators of degree $2^n - 1$; that is there are no generating manifolds of dimensions $2^n - 1$. In particular, $3 = 2^2 - 1$. This means that $\Omega_3^O = 0$ as well! Any closed, **unoriented** 3-manifolds is the boundary of some 4-manifold.

**Example 6.2.** The above game can be played with other types of cobordism. For example, maybe you want to consider unitary things instead of special orthogonal. It was shown that if we study $\Omega_*^U \otimes \mathbb{Q}$, the ring structure is $\mathbb{Z}[Y_2, Y_4, Y_6, ...]$ where the generators are in even degrees. Even more amazing, the generators can be taken to be $\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, ....$ If we forget the almost complex structure but remember orientation, we obtain $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Z}[Y_4, Y_8, Y_{12}, ...]$; all the $\mathbb{C}P^{2n+1}$ drop out which means that they are each the boundary of something.