Examples from Symplectic and Contact Geometry

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1 Symplectic Geometry

Recall that if M is a closed symplectic manifold, all its even Betti numbers are nonzero.

1.1 Examples of Symplectic Manifolds

Example 1.1. Let ω on $S^2 \subset \mathbb{R}^3$ be defined by $\omega_p(X, Y) = \langle p, X \times Y \rangle$. Here, we have the cross product and Euclidean dot product. In cylindrical coordinates with the radius fixed at $r = 1$, we can show that $\omega = d\theta \wedge dz$. Essentially, one notes that $a \cdot (b \times c)$ is the volume of a parallelpiped. Thus, ω is in fact the volume form of $\mathbb{R}^3$ contracted with p. Write $p = (x, y, z)$ as a vector field $W = x\partial_x + y\partial_y + z\partial_z$. Then $\omega = \iota_W d\mathrm{Vol}$.

This shows an ancient result of Archimedes. If we wrap the S^2 (radius 1) with a cylinder of radius 1, any band on the cylinder, when we project it onto S^2, orthogonal to the z-axis, the projected band has the same area as the band on the cylinder.

This is also a toy example of a moment map appearing in toric geometry. $S^2 = \mathbb{P}^1$ and it has $\mathbb{C}^* \subset \mathbb{P}^1$ as a dense open subset. Of course, $\mathbb{C}^*$ acts on itself and so $\mathbb{P}^1$ is toric. Then there is a map $\mu : \mathbb{P}^1 \to [-1, 1]$ where the preimage of points other than ± 1 are Lagrangian circles in S^2. The interval $[-1, 1]$ is called the Delzant polytope. This is quite an important example for SYZ fibrations and mirror symmetry.

Example 1.2. $\mathbb{C}P^n$ has symplectic structure. Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ and $\tilde{\omega}$ the restriction of the standard form ω_0 on $\mathbb{R}^{2n+2}$ to S^{2n+1}. Then $\tilde{\omega}$ is closed and invariant under $U(1) = S^1$ action on S^{2n+1}. Hence, it induces a closed 2-form on $\mathbb{C}P^n = S^{2n+1}/S^1$ which is nondegenerate.

Another way to construct this is by explicitly giving a formula. Consider $\tilde{\omega} = \frac{i}{2}\partial\bar{\partial} \log |z|^2$ on $\mathbb{C}^{n+1}$. One can see that $|z|^2$ is invariant under action by $U(n+1)$. The form descends to $\mathbb{C}P^n$ and is called the Fubini-Study form ω_{FS}. In a chart, say U_0, it equals $\frac{i}{2}\partial\bar{\partial} \log(1 + |z_1|^2 + ... + |z_n|^2)$. Note that by construction, we see that it is a real $(1, 1)$ form. It is not exact though of course, it is locally exact on the charts.

As a side note, consider a Kähler manifold X with any real smooth function $\phi : X \to \mathbb{R}$. Then $i\partial\bar{\partial}\phi$ is a real $(1, 1)$ form and is exact. This is easy to check as $i\partial\bar{\partial}$ is basically the Laplacian. Such a function ϕ is called a **Kähler potential** and a Kähler form can be described in this way locally. We may alter the Kähler form ω by adding $i\partial\bar{\partial}$ to it; this changes the Kähler metric but of course, such a deformation doesn't change the cohomology class of ω. For $\mathbb{C}P^n$, there is only one cohomology class of Kähler forms.

Example 1.3. Let M be any smooth n-manifold. There is a natural symplectic structure on its cotangent bundle $\pi : T^*M \to M$. Let $\xi \in T^*M$ and $\zeta \in T_\xi T^*M$. We define a 1-form λ in the following way: $\lambda_\xi(\zeta) = \xi(d\pi_\xi(\zeta))$. That is, $\lambda = \pi^*\xi$. Now let $(U, \varphi) = (U, x_1, ..., x_n)$ be a coordinate chart of M and $(\pi^{-1}(U), \bar{\varphi}) = (\pi^{-1}(U), \bar{x}_1, ..., \bar{x}_n, y^1, ..., y^n)$ be the induced chart on T^*M. Then, locally, $\xi = \sum y^i \, dx_i$ so

$$\lambda_\xi = \pi^*\xi = \pi^* \left(\sum y^i \, dx_i \right) = \sum y^i \, \pi^*(dx_i) = \sum y^i \, d(\pi^* x_i) = \sum y^i \, d\bar{x}_i.$$

Then it is apparent that $\omega := -d\lambda = \sum d\bar{x}_i \wedge dy^i$ which is locally, the symplectic structure on $(\mathbb{R}^{2n}, \omega_0)$.

1.2 Grassmannian of Linear Lagrangian Subspaces

Example 1.4. Let $LGr(n)$ represent the Grassmanian of linear Lagrangian n-planes in $(\mathbb{R}^{2n}, \omega_0)$. How might we compute the dimension of this space? Firstly, we may consider $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$. Then, consider homogeneous quadratic functions on $\mathbb{R}^n$. For example, when $n = 2$, we have $f(x, y) = ax_1^2 + bx_1x_2 + cx_2^2$. Then df is a 1-form but we consider it as a linear map and think of the image of df as a linear subspace in $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. All such linear spaces are transverse to the cotangent fiber and the defining coordinates are $(x_1, ..., x_n, 0, ..., 0)$. This means that ω_0 vanishes on these spaces. Thus, we have all the linear Lagrangian subspaces, save the cotangent fiber over 0. However, that is a single point and doesn't contribute to changing the dimension.

Therefore, we've reduced the problem of computing the dimension to basically computing the dimension of homogeneous quadratic forms which means computing the dimension of the space of symmetric $n \times n$ matrices. This is

$$\dim LGr(n) = n + \binom{n}{2} = \frac{n(n+1)}{2}.$$

In fact, we may use this to compute the dimension of the group of symplectic matrices on $\mathbb{R}^{2n}$, denoted $Sp(2n)$. It has dimension $2n(2n+1)/2 = n(2n+1)$.

Example 1.5. Another view of $LGr(n)$ is as follows. It is a classical fact that $U(n) \subset Sp(2n)$ acts transititvely on $LGr(n)$. Also, take $L = \mathbb{R}^n$, the first part of $\mathbb{R}^n \times \mathbb{R}^n$. L is Lagrangian and $O(n)$ maps $L = \mathbb{R}^n$ to itself. Thus, $LGr(n) \cong U(n)/O(n)$. The dimensions work out: $\dim U(n) = n^2, \dim O(n) = \binom{n}{2}$.

1.3 Lagrangian Submanifolds

Example 1.6. Consider the n-torus $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$. It has abelian group structure and under the involution $x \mapsto -x$, it has 2^n fixed points.

I believe that we may realize this in a symplectic way. Consider $\mathbb{C}P^n$ and $\mathbb{R}P^n$ embedded into it. The points of $\mathbb{R}P^n$ should have a representation of the form $[x_0 : ... : x_n]$ where these are all real entries. Also consider the Clifford torus T^n where it consists of points of the form $[z_0 : ... : z_n]$ where $|z_0| = |z_1| = ... = |z_n|$. Then the points may be represented by $[1 : e^{i\theta_1} : ... : e^{i\theta_n}]$. Both $\mathbb{R}P^n$ and T^n are Lagrangian submanifolds of $\mathbb{C}P^n$ and they intersect at exactly 2^n points: $[1 : \pm 1 : ... : \pm 1]$.

Example 1.7. We can always take a symplectomorphism $\varphi : (M), \omega \to (M, \omega)$ and consider its graph $\Gamma(\varphi) \subset (M \times M, \omega \oplus (-\omega))$. This is a Lagrangian submanifold in $M \times M$. In particular, id : $M \to M$ is a symplectomorphism and thus, the diagonal $\Delta \subset M \times M$ is Lagrangian.

Also, if we take the antipodal map $\alpha : S^2 \to S^2$, then the graph is a Lagrangian in $S^2 \times S^2$ with the symplectic form $\omega \oplus \omega$ (note the signs).

Example 1.8. Let M be any smooth manifold and consider it's cotangent bundle $\pi : T^*M \to M$. A point $(p, \varphi) \in T^*M$ comes from $p \in M$ and $\varphi \in T_p^*M$. We view T^*M as a manifold in its own right and define the Liouville 1-form λ on it in the following way. Let $p \in M, \varphi \in T_p^*M, v \in T_{(p,\varphi)}T^*M$. Then $\lambda_{(p,\varphi)}(v) = \varphi(d\pi_{(p,\varphi)}(v))$. If x^i form local coordinates on M around $p \in M$, then we have (x^i, ξ^i) forming local coordinates around $(p, \varphi) \in T^*M$ where φ looks like $\sum \xi^i dx^i$ locally, which is also the coordinate expression of λ. Thus, $\omega = -d\lambda = \sum dx^i \wedge d\xi^i$. This is clearly closed and is locally, symplectomorphic to $(\mathbb{R}^{2n}, \Omega)$ with its standard symplectic form. Thus, ω is a symplectic form.

Now, let η be a 1-form of M. We can view it as a section (and thus, a smooth map $M \to T^*M$) of $\pi : T^*M \to M$. Observe that $(\eta^*\lambda)_p(v) = \lambda_{\eta(p)}(d\eta_p(v)) = \eta_p(d\pi_{\eta(p)} \circ d\eta_p(v)) = \eta_p(v)$. Thus, $\eta^*\lambda = \eta$. This is the reason why λ is sometimes called the tautological 1-form.

Claim: $\eta : M \to T^*M$, thought of as a smooth map, is an embedding. It is a closed 1-form if and only if the image $\eta(M)$ is a Lagrangian submanifold of T^*M.

Proof. The fact that $\pi \circ \eta = \mathrm{id}_M$ means it is a smooth immersion and injective. We only need to show it is a proper map to show that it is an embedding. It is a basic fact that if X, Y are topological spaces and $f : X \to Y$ is continuous, then if Y is Hausdorff and f has a continuous left inverse, f is proper. Here, π is our continuous left inverse. Thus, η is an embedding.

To see the second statement, note that $\eta(M)$ is n-dimensional, being an embedding. Thus, we just need ω to vanish on $\eta(M)$ (be isotropic) or equivalently, $\eta^*\omega = 0$. But $\eta^*\omega = -\eta^*d\lambda = -d\eta^*\lambda = -d\eta$. Thus, η is a Lagrangian embedding if and only if $d\eta = 0$. □

Example 1.9. Consider the situation as above: M is a smooth manifold and its cotangent bundle is a symplectic manifold with a Liouville form: (T^*M, λ). Let $f : M \to \mathbb{R}$ be a Morse function and let L_0 be the zero section of T^*M while L_1 is the image of df. If we view df as a smooth map on manifolds, then we may pullback λ by df. Thus, we have $df^*\lambda = df$; i.e. λ restricted to L_1 is an exact 1-form. Therefore, ω on L_1 vanishes. L_1 is indeed, a Lagrangian submanifold since it is the image of a closed 1-form (see the example above), but moreover, it is the image of an exact 1-form. We call this an **exact** Lagrangian.

On the other hand, f is Morse so the image of df intersects the zero section L_0 transversely. L_0 is also an exact Lagrangian (just take any constant function g on M to find $0 = dg = \lambda|_{L_0}$). Let us scale f by a small $\epsilon > 0$ so that the two Lagrangian submanifolds are "close." Lastly, there is a Hamiltonian isotopy between them, generated by $\epsilon f \circ \pi$ where $\pi : T^*M \to M$ is the canonical projection.

It is a conjecture of Arnold that all compact exact Lagrangians of a cotangent bundle are Hamiltonian isotopic to the zero section. We see above that if the exact Lagrangian arises from a Morse function, this is doable. But in the general setting, this is an extremely difficult question. To date, the conjecture is only proved for T^*S^2 and $T^*\mathbb{R}P^2$ and uses 4-manifold theory. Fukaya, Seidel, and Smith have a paper showing that, if the base manifold M is a compact, simply-connected, spin manifold, and we assume that the exact Lagrangians under consideration are also spin, then they have the same homology. Mohammed Abouzaid has been able to drop the simply connected and spin conditions and even shown that they must be homotopically equivalent. But the conjecture remains quite elusive.

Example 1.10. In Paul Seidel's *Fukaya Categories and Picard-Lefschetz Theory*, he mentions in the introduction that the space of Lagrangian submanifolds is an infinite dimensional space. Locally around some $L \subset (M, \omega)$, this space is modeled on closed 1-forms on L. Up to Hamiltonian isotopy, the space locally has $b_1(L)$ degrees of freedom.

Sketch of proof: If we consider a loop space of a manifold M^n, locally around a loop $\gamma : S^1 \to M$, the loop space is modeled on sections of the bundle $\gamma^*TM \to S^1$. This is because there are n degrees of freedom to push the loop around, including the direction tangent to γ. This tangent direction is essentially a way to reparametrize the loop.

However, if we do not care about parametrization of the loop, we only look at normal directions. Similarly, if we're looking at Lagrangian submanifolds in M^{2n} to perturb, we only need to look at the normal directions. Thus, we should consider the normal bundle of L. Weinstein's neighborhood theorem says there is a neighborhood of L in M which is symplectomorphic to T^*L. Thus, if considering the space of submanifolds of dimension n we want to consider sections of $T^*L \to L$ which are 1-forms on L. However, it is the closed 1-forms which will give **Lagrangian** submanifolds, as discussed above.

Also from above, if we have an exact 1-form df where, say, f is Morse (recall, Morse functions are dense in $C^\infty(L)$), then $df(L)$ is an exact Lagrangian Hamiltonian isotopic to L_0. Thus, up to Hamiltonian isotopy, there are $b_1(L)$ degrees of freedom.

Example 1.11. (from Jonathan Evans' slides) Let $N \subset M^{2n}$ be a symplectic submanifold of codim 2 and suppose N has a Lagrangian submanifold L of dimension $n-1$. Since N is symplectic, it has a neighborhood which is symplectomorphic to a neighborhood of the zero-section in its symplectic normal bundle. We can use ω with a compatible J to define a metric and get a fixed-radius circle bundle of N. Restricting this to L will give an S^1-bundle over L inside of M. This bundle is Lagrangian in M.

Example 1.12. Let $\varphi : (M, \omega) \to (M, \omega)$ be an antisymplectic involution. This means that $\varphi^*\omega = -\omega$ and $\varphi^2 = \mathrm{id}$. The fixed point locus is isotropic. Recall, a submanifold is isotropic if ω vanishes when restricted to it. Since $\varphi_*^2 = \mathrm{id}$, then it has eigenvalues ± 1 and splits T_pM into two eigenspaces: $E_{\pm 1}$. It's easy to check that $\omega_p|_{E_1} = \omega_p|_{E_{-1}} = 0$. So both are isotropic which means that $\dim E_{\pm 1} \le n$. Since $T_pM = E_1 \oplus E_{-1}$, this means that they must both have dimension n. Another view point is to consider the operator $T = \mathrm{id} + \varphi_* : T_pM \to T_pM$. Its image is E_1, its kernel is E_{-1}. But the point is, if $V = v + \varphi_* v$, then $TV = V$, a fixed point of T. Through this discussion, one may hope to produce some Lagrangian submanifolds.

However, it is possible for an antisymplectic involution to lack fixed points. Take $\alpha : S^2 \to S^2$, the antipodal map. On the standard embedding of $S^2 \subset \mathbb{R}^3$, the symplectic form is $\omega = \iota_V vol$ where $V = x\partial_x + y\partial_y + z\partial_z$ and $vol = dx \wedge dy \wedge dz$. Then it is clear that $\alpha^2 = \mathrm{id}$ and $\alpha^*\omega = -\omega$. But α does not have any fixed points.

If we have a projective manifold, for example, then taking the real part will give a Lagrangian submanifold; e.g. $\mathbb{R}P^n \subset \mathbb{C}P^n$. This is because conjugation is an involution and is antisymplectic.

Example 1.13. Let $L \subset (M, \omega)$ be a Lagrangian and J a compatible almost complex structure. Let g be the Riemannian metric defined by ω, J. Observe that if $v, w \in TL$, then $g(Jv, w) = \omega(Jv, Jw) = \omega(v, w) = 0$. Thus, $JTL \perp TL$. This means that $J : TL \to TL^\perp \cong \nu_L$ is an isomorphism from the tangent bundle to normal bundle of L.

Example 1.14. Recall the Weinstein neighborhood theorem: Let $L \subset M$ be a Lagrangian. Then, there is a neighborhood of L symplectomorphic to a neighborhood of the zero section of T^*L.

This gives us the following lemma: Let $j : L \to M$ be a compact, orientable Lagrangian (so that $j_*[L]$ is a homology class). Then the self-intersection of $j_*[L]$ is $-\chi(L)$.

The proof is simple. Use Weinstein's neighborhood theorem to realize L as the zero section of T^*L. Then the self-intersection of the zero section is just the number of zeros of a generic 1-form, counted with signs. This is precisely $\chi(T^*L) = -\chi(TL) = -\chi(L)$.

As a corollary, the only compact, oriented Lagrangians in $\mathbb{C}^2$ are tori. This is because $j_*[L] \in H_2(\mathbb{C}^2, \mathbb{Z}) = 0$ and so $\chi(L) = 0$. Compact, oriented real surfaces are classified by Euler characteristic and so L must be a torus. In general, $j_*[L] \in H_n(\mathbb{C}^n, \mathbb{Z}) = 0$ so we need $\chi(L) = 0$.

However, it is a theorem of Gromov that $\mathbb{C}^n$ does not admit Lagrangian spheres. Indeed, he showed that if $L \subset \mathbb{C}^n$ is a compact, embedded Lagrangian, $H_1(L, \mathbb{R}) \neq 0$.

Example 1.15. Let $(M, d\alpha)$ be an exact symplectic manifold. Recall that a Lagrangian is exact if there is a function $f : L \to \mathbb{R}$ such that $df = \alpha|_L$. The function is unique up to a constant. We call a pair (L, f) an **exact Lagrangian brane** if L is an exact Lagrangian where $f : L \to \mathbb{R}$ is a choice of function such that $df = \alpha|_L$; it is called a **phase function**. This definition was specificed in Seidel's 2003 paper: *A long exact sequence for symplectic Floer homology.*

Here is a remark that Y. Oh makes about branes: The objects that occur as the natural boundary conditions of the given system in string theory roughly correspond to the notion of D-branes, which has been playing a fundamental role in the current string theory since around 1996 when the physicists introduced the concept of D-branes in open string theory.

Let's now consider a straightforward theorem:

Theorem 1.16. *Let $(M, d\alpha)$ be an exact symplectic manifold, and let (L, g), an exact Lagrangian brane with $i : L \to M$, be the inclusion. Then the Hamiltonian flow ϕ_t of some H induces a smooth family of exact Lagrangian branes (L_t, f_t) provided by $L_t = i_t(L)$, $i_t : L \to M$ and $f_t : L \to \mathbb{R}$, where*

$$i_t = \phi_t \circ i, f_t = g + \int_0^t (H_s + \alpha(X_{H_s})) \circ i_s \, ds.$$

Proof. Our goal is to show that $i_t^* \alpha = df_t$. $i_t^* \alpha = (\phi_t \circ i)^* \alpha = i^* \phi_t^* \alpha$. If we find the derivative of this, we can then integrate to obtain the original thing. What's the point of doing this? We can just focus on the time derivative of $\phi_t^* \alpha$.

$$\begin{aligned}
\frac{d}{dt} \phi_t^* \alpha &= \phi_t^* \mathcal{L}_{X_{H_t}} \alpha \\
&= \phi_t^* (d\iota_{X_{H_t}} \alpha + \iota_{X_{H_t}} d\alpha) \\
&= \phi_t^* (d\alpha(X_{H_t}) + \omega(X_{H_t}, \cdot)) \\
&= \phi_t^* (d\alpha(X_{H_t}) + dH_t) \\
&= d\phi_t^* (\alpha(X_{H_t}) + H_t) \\
&= d(\alpha(X_{H_t}) \circ \phi_t + H_t \circ \phi_t).
\end{aligned}$$

Now, let us set $i_0^* \alpha = dg$. Then

$$\begin{aligned}
i_t^* \alpha &= dg + \int_0^t \frac{d}{ds} i_s^* \alpha \, ds && (1.1) \\
&= d\left(g + \int_0^t (\alpha(X_{H_s}) \circ \phi_s + H_s \circ \phi_s) \circ i \, ds\right) && (1.2) \\
&= d\left(g + \int_0^t (\alpha(X_{H_s}) + H_s) \circ i_s \, ds\right) && (1.3)
\end{aligned}$$

This final expression on the right is precisely df_t. □

Observation: Even if $L_0 = L_1$, there is no guarantee that $g_1 = g_0$. This monodromy is an important feature to the Fukaya category of exact Lagrangian branes.

1.4 Symplectomorphisms vs. Hamiltonian Diffeomorphisms

Let (M, ω) be a symplectic manifold. Recall that the group of symplectomorphisms of M form a infinite dimensional group $\mathrm{Symp}(M, \omega)$ with Lie algebra consisting of vector fields X such that the 1-form $\iota_X\omega$ is **closed.**

$\varphi : M \to M$ is a Hamiltonian diffeomorphism if there exists a path of symplectomorphisms $\{\varphi_t\}_{0\leq t\leq 1}$ and a smooth function $H : [0, 1] \times M \to \mathbb{R}$ such that $\varphi_0 = \mathrm{id}_M$ and $\varphi_1 = \varphi$ and if X_t is the time-dependent vector field induced by the equation

$$\frac{d}{dt}\varphi_t = X_t \circ \varphi_t$$

then $\iota_{X_t}\omega = dH_t$. These form an infinite dimensional group called $\mathrm{Ham}(M, \omega)$ and the Lie algebra consists of vector fields X such that the 1-form $\iota_X\omega$ is **exact.** We may, of course, consider autonomous Hamiltonian diffeomorphisms; i.e. the X_t are independent of t.

Fact: $\mathrm{Ham}(M) \subset \mathrm{Symp}(M)$ is a normal subgroup. If $H^1(M, \mathbb{R}) = 0$, then $\mathrm{Ham}(M) = \mathrm{Symp}_0(M)$, the connected component of $\mathrm{Symp}(M)$ containing the identity.

If M is not closed, then it is possible for a Hamiltonian diffeomorphism to be fixed-point free.

Example 1.17. Consider translation in the x-direction of $\mathbb{R}^2$: $\varphi_t(x, y) = (x + t, y)$. This is a Hamiltonian diffeo because the function $H(x, y) = y$ is the Hamiltonian we seek; it is clearly fixed-point free.

Also, $dH = dy$ is translation invariant and thus, it descends onto the torus T^2. The translation maps (which might be better described as rotations on T^2), also descend and are still fixed-point free. Since ω on $\mathbb{R}^2$ is translation invariant, it descends to a translation invariant 2-form on T^2. Therefore, these φ_t are symplectomorphisms.

However, they are no longer Hamiltonian. The immediate reason is that dy is no longer exact; only locally exact. For there is no single chart with coordinate function y to give us this form. Along y-axis, dy descends to a 1-form of S^1 and it is a volume form; it cannot be exact.

This example demonstrates a more general phenomenon: on closed symplectic manifolds, an autonomous Hamiltonian diffeomorphism φ must have fixed points. After all, a function on a compact manifold has to have critical points. At a critical point x, $dH_x = 0$ which means, by the nondegeneracy of ω, our vector field $(X_H)_x = 0$ and thus, $\dot{\varphi}_t(x) = 0$. On the other hand, $\varphi_0(x) = \mathrm{id}(x) = x$. So $\varphi_t(x) = x$ and in particular, $\varphi(x) = \varphi_1(x) = x$.

1.5 Hamiltonian Floer Theory

Example 1.18. Here is a rather trivial example. Let's consider S^2 embedded in $\mathbb{R}^3$ in the usual manner with the standard complex structure J and symplectic form ω. Let $H : S^2 \to \mathbb{R}$ be the Morse function (our Hamiltonian) which is the usual height function (with respect to the z-axis): $H(x, y, z) = z$. Then the Hamiltonian vector field X is defined by $\iota_X\omega = -dH$; say we have also the standard metric $g := \omega(\cdot, J\cdot)$. Then $\omega(X, Y) = -dH(Y) = -\langle\nabla H, Y\rangle = -\omega(\nabla H, JY) = -\omega(Y, J\nabla H)$. So $X = J\nabla H$; as the gradient of H gives trajectories which are longitudinal lines down S^2, the trajectories of X are the lines of latitude. Thus, the flow generated by X are rotations of S^2 around the z-axis.

We can do some scaling of H so that the time 1 flow φ_1 is a rotation by 2π; i.e. $\varphi_1 = \mathrm{id}$. Then, we see that in this case, every point of S^2 is a fixed point of φ_1. Then this Hamiltonian is highly degenerate. However, in general, φ_1 only fixes the north and south poles and those are the only periodic orbits. The lower bound predicted by Arnold's conjecture is the sum of Betti numbers which is indeed 2.

Now, what if instead of $H(x, y, z) = z$, we had any smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $H(x, y, z) = f(z)$? Note that there is rotational symmetry around the z-axis. As such, it seems to me that considering this new Hamiltonian on S^2 is equivalent to considering the standard height function on a strangely embedded S^2 that has rotational symmetry about the z-axis.

Example 1.19. The purpose of this example is to show why we should **not** use the Morse index definition in an infinite dimension situation. It also demonstrates some basic computations from complex analysis. Let us consider $H \equiv 0$ on $M = \mathbb{C}$. The symplectic form is $\omega = dx \wedge dy = d\lambda$ where $\lambda = x\, dy$. In general, the action functional is

$$\mathcal{A}_H(\gamma) = -\int_{D^2} \overline{\gamma}^* \omega + \int_{S^1} H(t, \gamma(t))\, dt,$$

where $\overline{\gamma}$ is an extension of $\gamma : S^1 \to M$ to D^2. Since $H \equiv 0$, the second term is clearly 0. Since ω is exact on $\mathbb{C}$, then the first term is $-\int_{S^1} \gamma^* \lambda$. Note that $\frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy = \omega$ and $-\frac{i}{2} d(\bar{z} dz) = \omega$. Thus, in this situation

$$\mathcal{A}_H(\gamma) = \frac{i}{2} \int_\gamma \bar{z}\, dz.$$

Now let $\gamma(t)$ be a loop expressed in its Fourier expansion: $\sum_{k\in\mathbb{Z}} z_k e^{-ikt}$. This is contractible, being in $\mathbb{C}$. Then substituting,

$$\bar{z} = \sum_{k\in\mathbb{Z}} \bar{z}_k e^{ikt}, dz = -i\big(\sum_{k\in\mathbb{Z}} k z_k e^{-ikt}\big) dt.$$

Observe that taking the kth term in each, their product is $-ik|z_k|^2$. When $k \neq j$, we get two terms:

$$k z_k \bar{z}_j e^{i(k-j)t} + j \bar{z}_k z_j e^{i(j-k)t}.$$

However, the integral of $e^{i(k-j)t}$ from 0 to 2π is zero when $k \neq j$. This confirms the well-known fact in Fourier analysis that $\{e^{ikt}\}_{k\in\mathbb{Z}}$ forms an L^2 orthonormal basis. So we only care about the terms where $k = j$. This gives us that

$$\mathcal{A}_H(\gamma) = \pi \sum_{k\in\mathbb{Z}} k|z_k|^2.$$

If we have a Hermitian metric with some quadratic form A, then $\langle z, w\rangle = z^t A \bar{w}$. Letting $z = (z_k)_{k\in\mathbb{Z}}$, the action functional has infinite rank matrix A

$$A = \pi \begin{pmatrix} \ddots & & & & \\ & k-1 & & & \\ & & k & & \\ & & & k+1 & \\ & & & & \ddots \end{pmatrix}.$$

Here, the negative and positive eigenspaces both have infinite dimensions. Thus, the dimensions of the stable and unstable manifolds are infinite dimensional. We're not able to, using this index, find the dimension of the $\mathcal{M}(x, y)$ where x, y are critical points of $\mathcal{A}_H$.
Notice that this example also serves to illustrate that the action functional is neither bounded below nor above. For each $k \in \mathbb{Z}$ and letting $\gamma_k(t) := e^{ikt}$, $\mathcal{A}_H(\gamma_k) = k\pi$.

Example 1.20. Often, in Floer theory, we require some conditions on our symplectic manifold (M, ω) so that we can more easily do Floer theory or Gromov-Witten theory. For example, besides compactness (so we sometimes allow for manifolds with special boundary), we may ask for symplectic asphericity or strong symplectic asphericity. The first condition means that for every smooth map $f : S^2 \to M$,

$$\int_{S^2} f^*\omega = 0.$$

This is sometimes written as $\pi_2(\omega) = 0$. The second condition is that we have both $\pi_2(\omega) = 0$ and also, for all $f : S^2 \to M$,

$$\int_{S^2} f^*c_1 = 0.$$

This is sometimes denoted: $\langle c_1, \pi_2(M)\rangle = 0$. Put another way, the pullback bundle $f^*TM \to S^2$ is always sympletically trivial. It's clear that when $\pi_2(M) = 0$, then it is strongly symplectically aspherical. For any map $f : S^2 \to M$ is homotopic to a constant function and so there is a homotopy $H : S^2 \times I \to M$ such that $H(x, 0) = pt, H(x, 1) = f(x)$. Then

$$0 = \int_{S^2\times I} H^*d\omega = \int_{S^2} f^*\omega - \int_{S^2} pt^*\omega = \int_{S^2} f^*\omega.$$

Furthermore, $TM \to M$ is classified by the homotopy class of some map $v : M \to BSp(2n)$; $BSp(2n)$ deformation retracts to $BU(n)$. f^*TM is classified then by the homotopy class $f \circ v$ but $f \in \pi_2(M) = 0$ so then f^*TM is symplectically trivial.

One reason to assume strong symplectically asphericity is to avoid the phenomenon of bubbling (discussed later). Examples of such manifolds are Riemann surfaces of positive genus. The Uniformization Theorem, says that the universal cover of S^2 is itself, the universal cover of the torus is the complex plane, and the universal cover of higher genus Riemann surfaces is a disk. It's also the case that if $\widetilde{M} \to M$ is a covering, then $\pi_k(\widetilde{M}) \cong \pi_k(M)$ for all $k \geq 2$. Since $\mathbb{C}$ and the disk are contractible, then all their higher homotopy groups vanish and therefore, so do all the higher homotopy groups of positive genus compact Riemann surfaces.

Example 1.21. However, this strong symplectic asphericity is quite a restrictive condition. Thus, sometimes we ask instead that all smooth maps $f : S^2 \to M$ satisfy,

$$\int_{S^2} f^*c_1 = \lambda \int_{S^2} f^*\omega.$$

Here, there are three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. These roughly correspond to manifolds with positive curvature, are flat, and have negative curvature, respectively. The toy models are S^2, T^2, and Σ_g for $g \geq 2$.

The case of $\lambda > 0$ is called monotone. In real dim 4, a class of monotone symplectic manifolds comes from the the complex algebraic surfaces known as the del Pezzo surfaces. These are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{C}P^2$ blown up between 0 and 8 points, generically. Ono and Ohta proved that these are the only monotone symplectic 4-manifolds. More generally, there is a class complex projective varieties called Fano varieties which turn out to be monotone.

A complex manifold (X^n, J) is **Fano** if there is an **immersion** $\varphi : X \hookrightarrow \mathbb{C}P^N$ and a positive integer k so that $(K_X^{-1})^{\otimes k} = \varphi^*\mathcal{O}(1)$, where $K_X^{-1} = \Lambda^{n,0}TX$ is the dual of the canonical bundle. That's the definition for a line bundle to be ample. The immersion condition allows for some singular behavior.

Here is a theorem I read on the blog *The Electric Handle Slide*. The post was written by Nate Bottman.

Theorem: If (X, J) is a complex Fano manifold, then there exists a symplectic form ω on X so that (X, ω, J) is a monotone Kähler manifold. If (X, ω, J) is a projective monotone Kähler manifold, then it is Fano.

Let's consider why Fano implies monotone. Let (X^n, J) be Fano and $\varphi : X \hookrightarrow \mathbb{C}P^N$ our immersion. If ω_{FS} is the Fubini-Study form on $\mathbb{CP}^N$, then $c_1(T\mathbb{CP}^N) = [\omega_{FS}]$. Define $\omega_X := \varphi^*\omega_{FS}$. We show that (X, ω_X) is monotone.

ω_X is compatible with J because $\omega_X(Jv, JW) = \varphi^*\omega_{FS}(Jv, Jw) = \omega_{FS}(\varphi_*Jv, \varphi_*Jw)$. But φ is an immersion, which in this category, means it is holomorphic. So $\varphi_*J = \widetilde{J}\varphi_*$. Then $\omega_{FS}(\varphi_*Jv, \varphi_*Jw) = \omega_{FS}(\widetilde{J}\varphi_*v, \widetilde{J}\varphi_*w) = \omega_{FS}(\varphi_*v, \varphi_*w) = \omega_X(v, w)$. The penultimate equality comes from $\widetilde{J}$ being compatible with ω_{FS}.

Now, the 1st Chern class of a vector bundle equals the 1st Chern class of its top exterior power, so: $kc_1(TX) = kc_1(K_X^{-1}) = c_1((K_X^{-1})^{\otimes k}) = c_1(\varphi^*\mathcal{O}(1)) = [\omega_X]$. This fits the monotone condition, so long as ω_X is symplectic.

So we should check that ω_X is a symplectic form. It is closed since ω_{FS} is closed. The nondegeneracy of ω_X follows from the fact that if Y is a Kähler manifold and $Z \subset Y$ is a complex submanifold, then $\omega|_Z$ is nondegenerate. I think it should work even if Z is immersed and not embedded since there is no collapsing of tangent spaces. $\square$

The monotone-implies-Fano direction requires more machinery. It relies on the Nakai-Moishezon-Kleiman criterion which can be found in Rob Lazarsfeld's *Positivity I.*

Thus, we have a decent number of monotone manifolds from considering Fano manifolds. Among 3-folds, there are something like 108 Fano manifolds up to deformation.

By the way, in the case where $\lambda = 0$, this can be satisfied if $c_1 = 0$; e.g. in the case of Calabi-Yau manifolds which, having trivial canonical bundle means $c_1 = 0$. Note that for complex surfaces, these are the K3 surfaces.

1.6 Moment Map

See section starting on p. 161 of McDuff and Salamon's *Introduction to Symplectic Topology.*
Suppose we have a compact Lie group G which acts covariantly on a symplectic manifold M via symplectomorphisms. That is, there is a group morphism $G \to \mathrm{Symp}(M)$ such that $\psi_{gh} = \psi_g \circ \psi_h$ and $\psi_{\mathbb{1}} = \mathrm{id}$. The infinitesimal action defines a Lie algebra homomorphism $\mathfrak{g} \to \mathcal{X}(M, \omega) : \xi \mapsto X_\xi$:

$$X_\xi := \frac{d}{dt}\Big|_{t=0} \psi_{\exp(t\xi)}.$$

Since ψ_g is a symplectomorphism for all $g \in G$, then every X_ξ is a symplectic vector field; i.e. $\iota_{X_\xi}\omega$ is a closed 1-form. We say it is a Hamiltonian vector field if it is exact; i.e, there is some function H_ξ such that $dH_\xi = \iota_{X_\xi}\omega$. We call an action of G on M **weakly Hamiltonian** if every X_ξ is a Hamiltonian vector field. However, the H_ξ are unique up to a constant. We may choose a constant to make the map $\xi \mapsto H_\xi$ linear. The action is called **Hamiltonian** if the map

$$\mathfrak{g} \to C^\infty(M) : \xi \mapsto H_\xi$$

can be chosen to be a Lie group homomorphism with respect to the Lie algebra structure on $\mathfrak{g}$ and the Poisson structure on $C^\infty(M, \mathbb{R})$.

Now, assume the action of G is Hamiltonian on M. Then a **moment map** for the action is a map $\mu : M \to \mathfrak{g}^*$ such that the formula $H_\xi(p) = \langle \mu(p), \xi \rangle$ defines a Lie algebra morphism $\xi \mapsto H_\xi$ as above. In other words, given a Lie algebra morphism $\xi \mapsto H_\xi$ such that $X_\xi = X_{H_\xi}$ for all ξ, the map $\xi \mapsto H_\xi(p)$ is a linear functional on $\mathfrak{g}$ and it is denoted by $\mu(p)$.

Example 1.22. (Angular Momentum) Consider the diagonal action of $G = SO(3)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ with the standard symplectic form. So $\psi_\Phi(x, y) = (\Phi x, \Phi y)$ for $\Phi \in SO(3)$. By recalling that $\mathfrak{so}(3)$ consists of skew-symmetric matrices, i.e. $A^t = -A$, one can perform a straightforward calculation to see that the action is exact.

In fact, the action is generated by functions $H_A(x, y) = \langle y, Ax \rangle$ where $A \in \mathfrak{so}(3)$. We may also identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ via the map $\mathbb{R}^3 \to \mathfrak{so}(3) : \xi \mapsto A_\xi$,

$$A_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

Thus, $A_\xi x = \xi \times x$ for $x, \xi \in \mathbb{R}^3$ and

$$[A_\xi, A_\eta] = A_{\xi\times\eta}, \quad A_{\Phi\xi} = \Phi A_\xi \Phi^{-1}, \quad \mathrm{tr}(A_\xi^t A_\eta) = 2\langle \xi, \eta \rangle$$

The last identity implies that the standard inner product on $\mathbb{R}^3$ induces an invariant inner product on $\mathfrak{so}(3)$ and the dual $\mathfrak{so}(3)^*$ can be identified with $\mathfrak{so}(3)$ via this inner product. With this notation, the Hamiltonian function H_{A_ξ} can be written in the form $H_{A_\xi}(x, y) = \langle x \times y, \xi \rangle$. Hence, the moment map $\mu : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{so}(3)$ is given by $\mu(x, y) = A_{x\times y}$, where A is defined by the matrix above.

If x is the position and y the momentum, then $x \times y$ is the **angular momentum**. If $H : \mathbb{R}^6 \to \mathbb{R}$ is any Hamiltonian function which depends only on $|x|, |y|$ (such as motions in a central force field), then H is invariant under the action of $SO(3)$. This implies, in this case, the angular momentum determines three independent integrals of the motion.

1.7 Bubbling

Example 1.23. Consider the maps $u_n : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $z \mapsto (z, 1/(nz))$. Then, away from $z = 0$ (in the chart U_∞), as we let $n \to \infty$, the maps are converging to the curve $u : \mathbb{P}^1 \to \mathbb{P}^1 \times \{0\}$. but near $z = 0$, if we reparameterize the domain by the change of coordinates $z = 1/(nw)$, this converges to the sphere $\{0\} \times \mathbb{P}^1$. Thus, we get a bubble coming off of the first $\mathbb{P}^1$ and have something that appears as $\mathbb{P}^1 \vee \mathbb{P}^1$.

Or, take $u_n : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ where $z \mapsto (z, 1/(nz), 1/(n^2 z))$. Here, we get a bubble as before and an additional one at $\{0\} \times \{\infty\} \times \mathbb{P}^1$.

Example 1.24. An exercise in ch. 6 of Audin and Damian gives the example of a complex curve C in $\mathbb{C}^2$ defined by $y^2 = 4x^3 - x - 1$. Consider the map $C \to \mathbb{C}^2$ given by $(x, y) \mapsto (\alpha^2 x, \alpha^3 y)$ where $\alpha \in \mathbb{C}$. Then we can complete this map to a map $u_\alpha : C \to \mathbb{C}P^2$. We first homogenize the equation above to get $y^2 z = 4x^3 - xz^2 - z^3$. Then u_α sends $[x : y : z] \mapsto [\alpha^2 x : \alpha^3 y : z]$. Observe that if we let $X = \alpha^2 x, Y = \alpha^3 y, Z = z$, then the image of C satisfies the equation $Y^2 Z = 4X^3 - \alpha^4 X Z^2 - \alpha^6 Z^3$. This defines a "new" curve C' though it's really just C with a change of coordinates.

Observe that when $\alpha = 0$, then we're left with $Y^2 Z = 4X^3$, a cuspidal cubic. Letting $f = Y^2 Z - 4X^3$, we can see that it's singular:

$$\frac{\partial f}{\partial X} = -12X^2$$
$$\frac{\partial f}{\partial Y} = 2YZ$$
$$\frac{\partial f}{\partial Z} = Y^2$$

The point $[0 : 0 : 1]$ is on this curve and all the partial derivatives vanish at this point. Thus, it is a singular subvariety. However, the parametrized map $t \mapsto (t^2, \frac{1}{2}t^3)$ gives a curve satisfying $y^2 = 4x^3$. This is a smooth map; the differential vanishes as $t \mapsto 0$.

Here lies an important lesson: the **image** is singular but the **map** is smooth. Thus, if we look at u_α as $\alpha \to 0$, we should look at how the **maps** are converging, not what the **images** converge to. The topology on the space of maps is C^∞_{loc} so what it converges to is something rather strange. (This is the main lesson of bubbling to take away from this paragraph).

The family converges to a map which, when $z \neq 0$, sends $[x : y : z] \mapsto [0 : 0 : z] = [0 : 0 : 1]$. Thus, the curve C is sent to a point. However, these u_α are converging to a map v which has a different domain.

Consider what happens we if expand the domain to looking at $v : \mathbb{C}P^2 \to \mathbb{C}P^2$. We already know $C \mapsto [0 : 0 : 1]$. What if we look at points not in C, when $z = 0$ and $x \neq 0$? Then we have $[x : y : 0] \mapsto [\alpha^2 x : \alpha^3 y : 0] = [x : \alpha y : 0]$. Then with $\alpha = 0$, we have $[x : y : 0] \mapsto [1 : 0 : 0]$. That is, all of $S^2 = \mathbb{C}P^1 \mapsto [1 : 0 : 0]$ except for $[0 : 1 : 0]$. I'm not really confident in this reasoning and how $u_\alpha \to v$ but if this is correct, then the domain of v is C plus a $\mathbb{P}^1$.

2 Contact Geometry

Let M be a manifold of odd dimension $2n + 1$. Recall that a contact form θ is a nonvanishing smooth 1-form such that $d\theta$ is nondegenerate when restricted to $\ker \theta$ (which is rank $2n$). Observe that $d\theta$ is a symplectic tensor under this restriction. A contact structure on M is a smooth distribution ξ which is locally described by contact forms. If such a distribution exists, we call (M, ξ) a contact manifold.

Proposition 2.1. *A 1-form θ on M^{2n+1} is a contact form if and only if $\theta \wedge d\theta^n$ is nonvanishing on M.*

Note that this condition is the opposite of integrability in the sense of Fröbenius. A 1-form α defines an integrable distribution $\ker \alpha$ if and only if $\alpha \wedge d\alpha^n = 0$ everywhere. Thus, a contact structure is, in a sense, maximally non-integrable.

Example 2.2. Here are some standard contact structures:

1. On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, ..., x_n, y_1, ..., y_n, z)$, the standard contact form is
$$\theta = dz - \sum^n y_i dx_i.$$
Observe that $d\theta = \omega$, the standard symplectic form on $\mathbb{R}^{2n}$ (drop the last coordinate). Let us define some vector fields:
$$X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i}.$$
One can check that $\xi := \ker \theta = \text{span}\{X_i, Y_j\}_{i,j}$. Also, $d\theta(X_i, X_j) = d\theta(Y_i, Y_j) = 0$ and $d\theta(X_i, Y_j) = \delta_{ij}$. Thus, ξ is a contact structure defined by θ.

2. Let T^*M be the contangent bundle of any smooth manifold with Liouville form λ. Then $\mathbb{R} \times T^*M$ has contact form $\theta = dz - \lambda$ where z is the coordinate on $\mathbb{R}$.

Example 2.3. Let $\iota : S^{2n+1} \to \mathbb{R}^{2n+2}$ be the inclusion map. We define a 1-form on $\mathbb{R}^{2n+2}$:

$$\Theta = \sum^{n+1} x_i dy_i - y_i dx_i$$

and let $\theta = \iota^*\Theta$. If Ω is the standard symplectic form on $\mathbb{R}^{2n+2}$, observe that $d\Theta = 2\Omega$. Note that we can obtain Ω from taking d of $\sum^{n+1} x_i dy_i$ as well. So the "anti-derivative" is not unique up to constant, for forms. Now, define on $M = \mathbb{R}^{2n+2} - \{0\}$ the following vector fields:

$$N = \sum x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}, \quad T = \sum x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}.$$

With respect to the usual Euclidean Riemannian metric, N is normal to S^{2n+1} while T is tangent. Let $S = \operatorname{span}\{N, T\} \subset TM$ be a rank 2 subbundle and $S^\perp$ be its symplectic complement with respect to Ω. That is, $X \in S^\perp$ means $\Omega(N, X) = \Omega(T, X) = 0$.

Recall that for $p \in M$, S_p is a **symplectic vector space** if $S_p \cap S_p^\perp = \{0\}$. While this holds for orthogonal decompositions, it doesn't always hold for symplectic vector spaces. In our case, if $x \in S_p \cap S_p^\perp$, then for $y \in S_p$, $\Omega_p(x, y) = 0$ because $x \in S_p^\perp$. And if $y \in S_p^\perp$, $\Omega(x, y) = 0$ because $x \in S_p$. But $S_p + S_p^\perp = T_pM$ so, by nondegeneracy of Ω, $x = 0$. Therefore, S_p and $S_p^\perp$ are both symplectic vector spaces for all $p \in M$.

Also, observe the following contractions:

$$\iota_N d\Theta = 2\Theta, \quad \iota_T d\Theta = -2\sum^{n+1} x_i dx_i + y_i dy_i = -d(|x|^2 + |y|^2).$$

The function $f(x, y) = |x|^2 + |y|^2$ points radially outward and so any vectors normal to it; i.e. tangent to spheres, are in the kernel of df. That is $\ker df = TS^{2n+1}$.

Now, $d\Theta(N, T) = 2\Theta(T) = 2(|x|^2 + |y|^2) \neq 0$ on M. On the other hand, if $X \in S^\perp$, then $0 = d\Theta(N, X) = 2\Theta(X)$. So $X \in \ker\Theta$. On the other hand, $0 = d\Theta(T, X) = -df(X)$ means $X \in \ker df = TS^{2n+1}$. What we've shown here is that $S^\perp \subset \ker\Theta \cap TS^{2n+1}$. $S^\perp$ is rank $2n$ while the two bundles on the right are each of rank $2n+1$. However, observe that $0 = d\Theta(N, N) = 2\Theta(N)$ so $N \in \ker\Theta$ but $N \notin TS^{2n+1}$. Thus, the by dimension reasons, $S^\perp = \ker\Theta \cap TS^{2n+1}$. Moreover, $\theta = \Theta|_{S^{2n+1}}$ so $S^\perp = \ker\theta$.

Lastly, $d\theta = \iota^* d\Theta = d\Theta|_{S^{2n+1}}$. Since $\ker\theta \subset TS^{2n+1}$, restricted to $S^\perp$, $d\theta = d\Theta$. If we fix $X \in S^\perp$ and find that $d\Theta(X, Y) = 0$ for all $Y \in S^\perp$, then $X = 0$. This is because the only elements symplectically complement to all of $S^\perp$ are linear combinations of N and T, neither of which reside in $S^\perp$. Thus, $X = 0$ is the trivial linear combination. This shows that $d\theta$ is nondegenerate on $\ker\theta$ and hence, is a contact form with distribution $\xi = S^\perp = \ker\theta$.

Theorem 2.4. *Let (M, ξ) be a contact manifold with defining contact form θ. There exists a unique vector field X such that $\iota_X d\theta = 0$ and $\theta(X) \equiv 1$. This vector field is called the Reeb field.*

Proof. Let $\Phi : TM \to T^*M$ be a bundle morphism which sends $V \mapsto \iota_V d\theta$. For $p \in M$, if we restrict the map $\Phi_p : T_pM \to T_p^*M$ to ξ_p, then it is injective because $d\theta_p|_{\xi_p}$ is nondegenerate. Then, Φ_p has rank at least $2n$. If it had rank $2n+1$, this implies $d\theta_p$ is nondegenerate on T_pM which is not possible as $\dim T_pM$ is odd. So the rank of Φ_p is $2n$ and its kernel is 1 dimensional.

This means $\ker\Phi_p$ is not contained in $\xi_p = \ker\theta_p$. Then, there is a unique vector $X_p \in \ker\Phi_p$ such that $\theta_p(X_p) = 1$. And of course $0 = \Phi_p(X_p) = d\theta_p(X_p, -)$. Thus, we've found a vector field that satisfies the properties above. We only need to show smoothness which follows because $\ker\Phi$ is a smooth distribution. $\square$

If X is a Reeb field for contact form θ, note that $\mathcal{L}_X\theta = d\iota_X\theta + \iota_X d\theta$. The second term is 0 by the properties of the Reeb field and $\iota_X\theta \equiv 1$ so $d(1) = 0$. Thus, $\mathcal{L}_X\theta = 0$.