SUSY & Morse Theory

Outline:

1. Brief Intro to Supersymmetry
2. Morse Theory
3. SUSY Quantum Mechanics
4. SUSY Quantum Field Theory

Comment: M. Atiyah was asked to write something to summarize Edward Wittens' work around the time Witten was awarded the Fields Medal in 1990

1. Introduction to SUSY & QFT

In any QFT, there is a Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$

bosons $\mathcal{H}^+$, fermions $\mathcal{H}^-$

Bosons: force-carrying particles, integer spin

Fermions: matter particles, half-integer spin, obeys Pauli exclusion principle

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SUSY must have (Hermitian) symmetry operators

Q_i, $i=1,\dots,N$ which map $\mathcal{H}^\pm \to \mathcal{H}^\mp$

def: $(-1)^F$ will be the operator $(-1)^F\big|_{\mathcal{H}^\pm} = \pm \mathrm{Id}$

Basic Conditions of SUSY

1. $(-1)^F Q_i + Q_i (-1)^F = 0$. The Q_i are odd
2. If H is the Hamiltonian, then

$$Q_i H - H Q_i = 0$$

Note: H generates time translations (it gives the dynamics)

3. $Q_i^2 = H$

4. For $i \neq j$, $Q_i Q_j + Q_j Q_i = 0$.

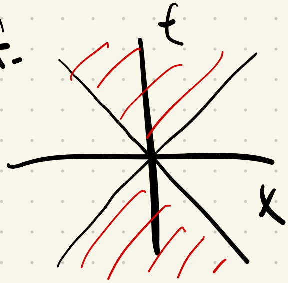
In relativistic settings, we have Lorentz transformations which intermingle space & time translations. In this case, we have further conditions.

Take the simplest case: 1+1 spacetime; has only one momentum operator P.

In this situation, there are only two symmetry operators: Q_1 & Q_2 & they satisfy

5. $Q_1^2 = H+P$, $Q_2^2 = H-P$, $Q_1 Q_2 + Q_2 Q_1 = 0$

cf.



light cone

when $x \pm t = 0$, we're on the boundary of the light cone.

Rmk: symmetric means it commutes w/ the Hamiltonian H.

(5) + Jacobi identity gives

6. $[Q_i, H] = [Q_i, P] = 0$

Observe that (5) also gives: $H = \frac{1}{2}(Q_1^2 + Q_2^2)$
which is positive semi-definite (Q_1, Q_2 are Hermitian)

Since the Q_i are odd, then H & P are even &
$[H, (-1)^F] = [P, (-1)^F] = 0$.

Most Important Question about a SUSY theory

Does there exist a state $|\Omega\rangle$ s.t. $Q_i|\Omega\rangle = 0$ for each i? (*)

If $\exists\, |\Omega\rangle$, then $H|\Omega\rangle = 0 \Rightarrow$ it has energy $= 0$.

Thus, $|\Omega\rangle$ is a vacuum state; i.e. a state of minimum energy.

Rmk: The number of such solutions to (*) is not as important as whether there are any.

Assuming SUSY:

$\exists$ solution to $(*)$ $\Rightarrow$ bosons & fermions have equal mass.

In our experiments, bosons & fermions do not have equal mass.

Thus, if SUSY is true in our universe, there is no such solution & vacuum states must have positive energy.

In such a world where $(*)$ has no solutions we say <u>supersymmetry</u> is spontaneously broken.

It is very difficult in general to show if $(*)$ has solutions. It is akin to showing the Dirac operator on a cpt mfd has a zero eigenvalue.

$\therefore$ Indirect methods may be better.

Note: $Q_i|\Omega\rangle = 0 \Rightarrow P|\Omega\rangle = 0$. So restrict attention to $\mathcal{H}_0$ = 0-eigenspace of P. A state in $\mathcal{H}_0$, if annihilated by one Q_i $\Rightarrow$ it is annihil by every Q_i.

Choose one; call it Q.

We frame this now as an index problem:

Since $\mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathcal{H}_0^-$, decompose $Q = Q_+ + Q_-$.

We have $Q_+ : \mathcal{H}_0^+ \to \mathcal{H}_0^-$ & $Q_- = Q_+^*$ (adjoint)

If $\operatorname{Ind} Q_+ \doteq \dim \ker Q_+ - \dim \ker Q_- \neq 0$, then

Q has a zero eigenvalue in $\mathcal{H}_0$.

Claim 1: $\operatorname{Ind} Q_+ = \operatorname{Tr}(-1)^F$.

Claim 2: There are SUSY theories w/ $\operatorname{Ind} Q_+ \neq 0$ &

$\therefore$ there is no spontaneous symmetry breaking.

Further Remarks on SUSY: & so they come in pairs.

The idea is we can interchange bosons & fermions. It has never been observed.

But it gives localization: we have some complicated integral over an ∞-dim space of commuting & anti-commuting fields.

SUSY says we're integrating something like an exact differential form & so the integral localizes at the critical pts.

This reduces the integral over a fin-dim moduli space.

e.g. instantons, algebraic curves (Donaldson, Gromov-Witten)

2. Morse Theory (Simplest SUSY QM system)

Let (M, g) be a Riem mfld, d, d^* the exterior derivative & its adjoint

Let $Q_1 = d + d^*$, $Q_2 = i(d - d^*)$, $H = \Delta \doteq dd^* + d^*d$

It's easy to check: $Q_1^2 = Q_2^2 = H$, $Q_1 Q_2 + Q_2 Q_1 = 0$.

$\Omega^p = \{p\text{-forms}\}$, p-even = bosons
p-odd = fermions

Let $h : M \to \mathbb{R}$ & $t \in \mathbb{R}$. Define

Note the signs

$$d_t = e^{-ht} d\, e^{ht}, \quad d_t^* = e^{ht} d^* e^{-ht}$$

multiplication by e^{ht} operator

If we let $Q_{1t} = d_t + d_t^*$, $Q_{2t} = i(d_t - d_t^*)$

$$H_t = d_t d_t^* + d_t^* d_t,$$

we have something just as above.

Let α be a diff form, $\{a^k\}$ an ONB of tangent vectors at $p \in M$.

a^k can be viewed as an operator on $\Lambda^* T_pM$ by interior multiplication

$a^k(\varphi) = \iota_{a^k}\varphi$. The dual operator a^{k*} is operation by wedge
(create)
(annihilate)

$$d_t\alpha = e^{-th} d(e^{th}\alpha) = e^{-th}(te^{th}\, dh \wedge \alpha + e^{th} d\alpha)$$

$$= t\, dh \wedge \alpha + d\alpha.$$

$$\therefore d_t = d + t \sum_i \frac{\partial h}{\partial \phi^i} a^{*i} \quad \text{(locally)}$$

Similarly, $d_t^* = d^* + t \sum_i \dfrac{\partial h}{\partial \phi^i} a^i$ (locally).

This helps us compute

$$H_t = \Delta + \underbrace{t^2 |\nabla h|^2} + t \sum_{i,j} \frac{\partial^2 h}{\partial \phi^i \partial \phi^j} [a^{*i}, a^j].$$

We'll see later that this term is very important & represents the potential energy.

def. $B_p(t)$ = Betti # w/ d_t : dim of space of d_t-closed p-forms which are not d_t-exact

Claim: $B_p(t) = B_p \doteq B_p(0)$ ← the usual Betti number

pf: d_t is just d conjugated by e^{ht}, which is an invertible operator. So $\psi \longmapsto e^{-ht}\psi$ is an invertible mapping from closed but not exact p-forms to d_t closed but not d_t exact p-forms. $\boxed{}$

Moreover, the number of harmonic p-forms in the sense of H_t equals B_p.

Rmk: This independence of t is useful b/c as $t \to \infty$, the spectrum of H_t simplifies. We will then place upper bounds on B_p using crit pts of h.

How does h enter into H_t? Let $\{a^k(p)\}$ be an ONB of T_pM.

Regard it as an operator on $\Lambda^* T_pM$: $\psi \longmapsto \iota_{a^k}\psi$ (interior mult.)

Then a^{k*} is the adjoint: $\psi \longmapsto A^k \wedge \psi$

↑dual 1-form to a^k.

Rmk: In physics literature,

a^{k*} = fermion creation operator b/c it increases degree w/ wedge

a^k = fermion annihilation operator b/c it decrease degree

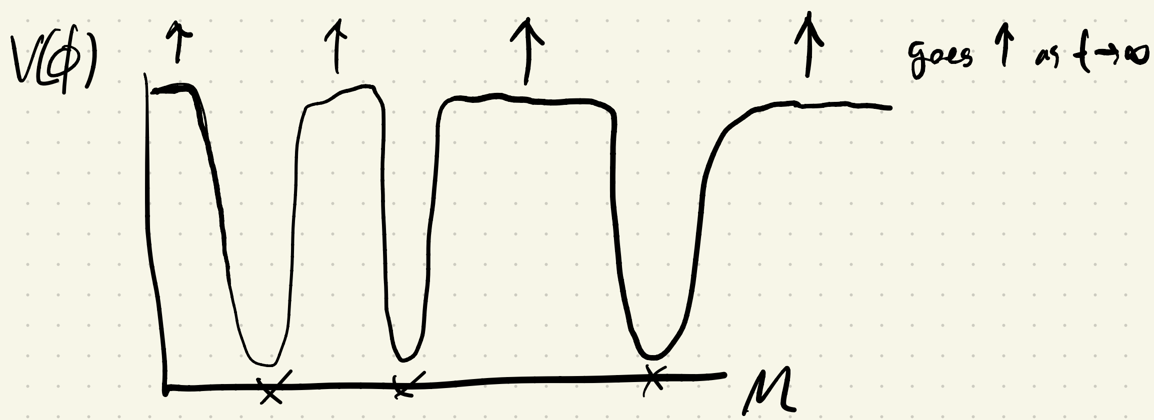
On a Riem mfd, it makes sense to take a covariant 2^{nd} derivative on the dual basis to a^k. $\circ \atop {2\,0}$

$$H_t = \underbrace{dd^* + d^*d}_{\text{Laplacian}} + \underbrace{t^2 |Dh|^2}_{\text{potential}} + \underbrace{t \sum_{i,j} \frac{D^2 h}{D\phi^i D\phi^j} [a^{i*}, a^j]}_{\text{other stuff}}$$

$$|Dh|^2 = g^{ij} \left(\frac{\partial h}{\partial \phi^i}\right)\left(\frac{\partial h}{\partial \phi^j}\right)$$

As $t \to \infty$, $V(\phi) \doteq t^2 (dh)^2$ becomes large away from the crit pts of h, when $dh = 0$.

Thus, eigenfunctions of H_t are concentrated near the crit pts.



So the eigenf^ns approach sums of Dirac-delta f^ns.

This alludes to the localization idea from earlier.

Claim: Asymptotic expansion of the eigenvalues in powers of $1/t$ can be calculated w/ local data around crit pts.

Let $h: M \to \mathbb{R}$ be Morse w/ crit pts p^a. The Hessian

$$\frac{D^2 h}{D\phi^i D\phi^j} \text{ is nonsingular}$$

Let $M_p = \#$ crit pts w/ Morse index p.

Prop. $M_p \geq B_p$ (Morse Inequalities)

Steps of proof:

1. Using perturbation theory ideas, approximate H_t near crit pts. by an operator $\overline{H_t}$.

2. Compute the spectrum of $\overline{H_p}$ & conclude that to every crit pt of h, there is only one eigenstate of $\overline{H_t}$ whose energy does not diverge w/ t.

3. Not each eigenstate of $\overline{H_t}$ is an eigenstate of H_t but the converse is true.

$\therefore M_p \geq B_p$.

In more details:

Let $\lambda_p^{(n)}(t)$ be the n^{th} smallest eigenval of H_t:

$$\lambda_p^{(n)}(t) = t\left(A_p^{(n)} + \frac{B_p^{(n)}}{t} + \frac{C_p^{(n)}}{t^2} + \cdots\right).$$

Of course, $B_p = \#\{n \in \mathbb{N} : \lambda_p^{(n)}(t) = 0\}$. Also, for large t,

$$\#\{n \in \mathbb{N} : \lambda_p^{(n)}(t) = 0\} \leq \#\{n \in \mathbb{N} : A_p^{(n)} = 0\}$$

It suffices to show: The RHS $= M_p$.

Let ϕ_i be coord s.t. at crit pts p^a, $\phi_i = 0$. Then near p^a,

$$h(\phi_i) = h(0) + \frac{1}{2}\sum \lambda_i \phi_i^2 + O(\phi^3) \text{ for some } \lambda_i.$$

We approx H_t near p^a w/

$$\overline{H}_t = \sum_i \Big(\underbrace{\underbrace{-\frac{\partial^2}{\partial \phi_i^2}}_{\text{Laplacian}} + \underbrace{t^2 \lambda_i^2 \phi_i^2}_{\text{potential}}}_{\cong\, H_i} + t\lambda_i \underbrace{[a^{i*}, a^i]}_{\cong\, K_i} \Big)$$

other stuff

The correction terms $O(\phi^3)$ can be ignored if we only wish to compute $A_p^{(n)}$ [again relying on the eigenf^{ns} to concentrate at crit pts as $t \to \infty$)

So $\overline{H}_t = \sum (H_i + t\lambda_i K_i)$

Claims: ∘ H_i, K_j mutually commute } so can be simultaneously diagonalized.

∘ H_i is the simple harmonic oscillator whose eigenval are well-known: $t|\lambda_i|(1+2N_i)$, $N_i = 0, 1, 2, \ldots$ These appear w/ multiplicity 1.

∘ Note that the eigenf^{ns} of $\overline{H}_i$ vanish rapidly if $|\lambda_i \phi_i| \gg 1/\sqrt{t}$ $\Rightarrow$ the approx $\overline{H}_t$ is valid to lowest order in $1/t$.

∘ K_j has eigenval ± 1

Then, the eigenval of H_t are:

(**) $$t \sum_i \left(|\lambda_i|(1+2N_i) + \lambda_i \varepsilon_i \right), \quad N_i = 0, 1, 2, \ldots, \quad \varepsilon_i = \pm 1$$

Witten says if we restrict $H_t|_{\Omega^p}$, then the # of positive ε_i for the K_i's must be p. Not sure why

For (**) to vanish we need all the $N_i = 0$; $\varepsilon_i = +1$ iff $\lambda_i < 0$.

$\therefore$ Around any crit pt, H_t has exactly one zero eigenf^n which is a p-form if the crit pt has Morse index p.

The other eigenval are proportional to t w/ positive coeff.

Then (**) explicitly gives $A_p^{(n)}$ in the spectrum of H_t near p^a.

$\Rightarrow$ For each crit pt, H_t has exactly one eigenstate $|a\rangle$ whose energy does not diverge w/ $t \to \infty$. $|a\rangle \in \Omega^p$ if the assoc. crit pt has index $= p$.

Now, H_t doesn't annihilate all of these $|a\rangle$, just the leading $A_p^{(n)}$ terms. But H_t does not annihilate any other states b/c they have energy proportional to t for large t.

$\therefore$ $\#\{\text{zero energy } p\text{-forms}\} = B_p \leq M_p$. ∎

Rmk: This shows that there is a 1-1 correspondence

$\{$ w/ states $|a\rangle$ s.t. $\bar{H}_t|a\rangle=0$ $\}$ $\leftrightarrow$ crit pts of h.

Since $\bar{H}_t \approx Q_t^2 = (d_t + d_t^*)^2$, then approx. either

$|a\rangle \in \ker Q_t$ or $Q_t|a\rangle \in \ker Q_t$.

Thus we've found some approx. solutions to:

$$Q_{1t}|a\rangle = Q_{2t}|b\rangle = 0.$$ in this simple case of 2 symmetric operators

This means that the number of SUSY vacua is bounded below by $\sum_p B_p$ $\leftarrow$ a topological invariant of M!

The strong Morse Inequalities.

$$\sum M_p t^p - \sum B_p t^p = (1+t) \sum \overset{\mathbb{Z}_{\geq 0}}{\overset{\in}{Q_p}} t^p$$

This eqn is equiv to the assertion that the crit pts model the (co)homology of the mfd M.

It's saying that the difference on the LHS has a "positive" leftover bit. eg. $M_p - B_p = Q_p + \underbrace{Q_{p-1}}$ some exact things shifted up by a boundary operator ∂.

We already have our (co)boundary operator; it's the d_f we saw from before.

Witten goes on to attempt refining these Morse inequalities. We obtained the inequalities through an approx calculation of the spectrum of H_t. A more accurate calc. could give better bounds.

It's tempting to try computing the higher terms like $B_p^{(n)}, C_p^{(n)}, \ldots$

However, if $A_p^{(n)}$ vanishes, then these higher terms vanish in $1/t$. I'm not clear on this explanation. He says the higher terms are computable w/ local data; so we don't know if the existence of a crit pt is dictated by global topology or if it is "removable."

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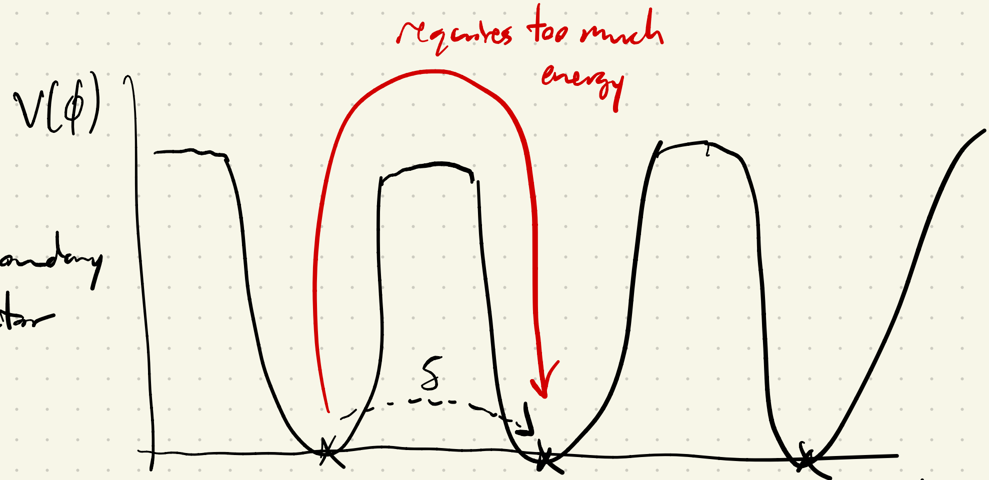
ie to gain new info, we study something sensitive to the existence of multiple crit pts. A good candidate is

$$V(\phi) = t^2 |Dh|^2 \quad (\text{it has a minimum for each crit pt})$$

Witten interprets the flow lines of Dh & the boundary operator in terms of tunneling (or instanton corrections)

Rmk. His reference for instanton corrections is Milnor's "Lectures on h-cobordism." Typo?

Notation: X_p = $\mathbb{R}$-vect space generated by index p crit pts



δ = (co) boundary operator

The way written assigns orientation to flow lines is interesting.

At a crit pt A, there is a state $|\alpha\rangle$ of ≈ 0 energy.

Suppose $|\alpha\rangle$ is a p-form. Then let V_A = vect space spanned by negative eigenvectors of $\frac{D^2 h}{D\phi^i D\phi^j}$ at A.

$\dim V_A = p$. Let Γ be a flow line from B (ind $= p+1$) to A.

Let v be the tangent vect of Γ at B & $\tilde{V}_B = \langle v\rangle^\perp$ in V_B.

Orientation of $\tilde{V}_B$ is inherited from V_B & flow lines near Γ give mapping $\tilde{V}_B \xrightarrow{f} V_A$. Let $n_\Gamma = \begin{cases} +1, & f \text{ preserves orient} \\ -1, & f \text{ reverses orient} \end{cases}$

Of course, $n(a,b) = \sum_{\Gamma} n_{\Gamma}$; $\delta|a\rangle = \sum_{b} n(a,b)\cdot|b\rangle$

Instanton calculations show that states <u>not</u> annihilated by $\Delta_\delta \doteq \delta\delta^* + \delta^*\delta$ do <u>not</u> have zero energy. In fact, for large t, the energy is roughly

$$\exp\left(-2t\,|h(A)-h(B)|\right).$$

Let $Y_p = \#\{0\text{-eigenstates of } \Delta_\delta \text{ acting on } X_p\}$

We see that $B_p \le Y_p$. Does $Y_p = M_p$?

One cannot answer this based on instanton considerations b/c some non zero energy states may be at approx zero energy & is undetected as nonzero using perturbation theory techniques & instanton calculations. The energy decays more rapidly than $\exp(-2t\,|h(A)-h(B)|)$.

Derivation of $\delta|a\rangle = \sum_b n(a,b) \cdot |b\rangle$

The system described by d_t, d_t^*, H_t can be obtained by canonical quantization of a Lagrangian $\mathcal{L}$ (complicated)

I wonder if this is some equivalence b/w Hamiltonian & Lagrangian formalism, the equiv. furnished by a Legendrian.

$\mathcal{L}$ has terms curvature terms & also seems to have a time coordinate λ.

So we're in a $(\dim M)+1$ spacetime?

I think Witten discards the fermionic terms in $\mathcal{L}$ & assumes the manifold is flat (curvature terms vanish) in order to write a new action:

$$I = \frac{1}{2}\int \left(g_{ij} \frac{d\phi^i}{d\lambda}\frac{d\phi^j}{d\lambda} + t^2 g^{ij} \frac{\partial h}{\partial \phi^i}\frac{\partial h}{\partial \phi^j}\right) d\lambda$$

metric

The crit pts of $\mathcal{I}$ are the instanton solutions, aka the tunneling paths or flow lines.

Via manipulations:

$$\mathcal{I} = \underbrace{\frac{1}{2}\int \left| \frac{d\phi^i}{d\lambda} \pm t g^{ij} \frac{\partial h}{\partial \phi^j} \right|^2 d\lambda}_{\geq 0} \mp \overbrace{t \int \frac{dh}{d\lambda} d\lambda}^{t\,(h(\infty) - h(-\infty))}$$

$$\Rightarrow \mathcal{I} \geq t \left| h(\infty) - h(-\infty) \right| \quad \text{(take limits)}$$

& there's equality iff

$$\frac{d\phi^i}{d\lambda} \pm t g^{ij} \frac{\partial h}{\partial \phi^j} = 0$$

Thus, if Γ is a flow line b/w crit pts B & A, then its action is

$$I \doteq \mathcal{I}(\Gamma) = t\,|h(B) - h(A)|.$$

The instanton contributions to H_E are of the order $\exp(-2I)$ which explains why studying instantons cannot answer whether $Y_p = M_p$. (See two pages back)

Rmk. Apparently, when calculating instanton corrections, the next step is usually be the evaluation of the Fredholm determinant for small fluctuations about the classical solution. But the nonzero eigenval for bosons & fermions cancel due to SUSY. $\therefore$ we only have zero eigenval of fermions left.

Let Γ be a trajectory b/w A & B. Then

$$\underbrace{\mathrm{Ind}_\Gamma \not{D}}_{\text{Fredholm index of Dirac operator linearized at } \Gamma} = \underbrace{\mathrm{Ind}(A) - \mathrm{Ind}(B)}_{\text{Morse indices}}$$

We want to study the cases where the Dirac operator has exactly one 0-eigenvector (aka zero mode or harmonic spinor). In that case $\dim \ker \not{D} = 1$ & possibly,

$$\mathrm{Ind}_\Gamma \not{D} = 1 = \mathrm{Ind}(A) - \mathrm{Ind}(B).$$

In Morse theory, we care about indices differing by 1.

Rmk: Witten gives a physical reason for studying the case where the Dirac operator has exactly one zero mode: it lets us evaluate the action of d_f on very low energy states. & it's relevant that d_f is linear on fermion fields, apparently. As a byproduct, we have the Morse theoretic reasons.

Of course, if the trajectories b/w A & B (index differing by 1) are isolated, then $\not{D}$ has exactly one zero mode & it can be calculated from the classical solution by a SUSY transformation

Witten says:

the normalization may be the bosonic zero mode...

"The normalization factor associated with the fermion zero mode cancels in magnitude against the normalization factor associated with the fact that our classical solution is really a 1-parameter family of solutions" (because any solution is still a solution under translation).

I'm not sure what he means. The second part seems to be about quotienting the space of solutions by $\mathbb{R}$ to get a moduli space of trajectories: $\mathcal{M}(A,B)$

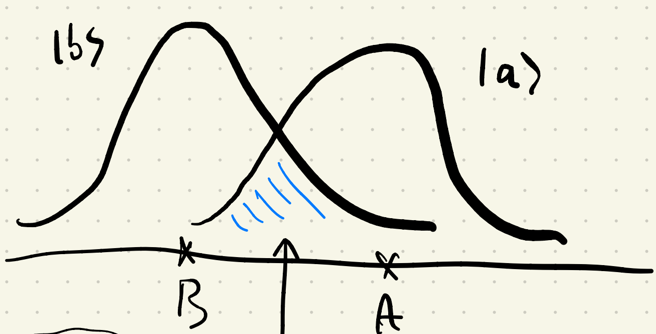
quotienting is the normalization (?)

Perhaps he means: $\dim\left[(\ker \not{D}_1)/\mathbb{R}\right] = 0 = \dim \mathcal{M}(A,B)$

Claim: Let $|a\rangle, |b\rangle$ be eigenstates associated to crit pts A & B & Γ is a flow line b/w A & B. Then, the amplitude $\langle b, d_t a\rangle$ of Γ is $\exp(-t\,|h(B)-h(A)|)$

What is amplitude? I don't quite know in physics terms but the eigenstates $|a\rangle$ & $|b\rangle$ concentrate around A & B, resp. So they decay rapidly away from A & B, resp.

However, the decay rate is slowest along trajectories like Γ & the decay rate is $\exp(-t\,h(\phi))$ which looks like $\exp(-t\,|h(B)-h(A)|)$.



Seems Γ is the path of steepest decent so is the fastest path from B to A considering ∇h. However, it is the slowest path of decay for $|b\rangle$ & $|a\rangle$.

overlap is greatest along traj Γ.

Before, we discussed how to give signs to Γ. The physical interpretation seems to use Γ as a propagator of state $|b\rangle$ to $|a\rangle$ & gives the sign of n_Γ based on the sign of the amplitude $\langle b | d_t a \rangle$. This is the WKB approach.

This discussion suggests that the boundary operator is

$$\tilde{\partial}|a\rangle = \sum_b e^{-t(h(B)-h(A))} n(a,b) \cdot |b\rangle.$$

I think the amplitude $\langle b | d_t a \rangle = \sum_\Gamma n_\Gamma e^{-t(h(B)-h(A))}$

However, we can just undo the conjugation by e^{th} in d_t.

The e^{th} don't carry any info in defining $\tilde{\partial}$.

This gives δ which is just a rescaling of $\tilde{\partial}$.

But then $\tilde{\partial}^2 = 0 \iff \delta^2 = 0$. & we know that

$\delta = d_t$ for large t. So $\tilde{\partial}^2 = 0 \iff \delta^2 = 0$
$\iff d_t^2 = 0$

$$\langle b | d_t a \rangle \doteq \int_M \langle b, d_t a \rangle \, d\mathrm{Vol}$$

$$= \int_M b \wedge * d_t a$$

$$= \int_M b \wedge * (da + t\, dh \wedge a)$$

The claim is that

$$\langle b | d_t a \rangle = \sum_{\Gamma} n_{\Gamma} e^{-t(h(B) - h(a))} \quad \text{as well}$$

$$= n(a,b)\, e^{-t(h(B) - h(A))}$$

So $\hat{\partial} |a\rangle = \sum_{|b\rangle} \langle b | d_t a \rangle \cdot |b\rangle$

(this is a bit confusing w/ how $B \xrightarrow{\Gamma} A$

So $\hat{\partial}^2 |a\rangle = \sum_{|b\rangle} \sum_{|c\rangle} \left(\langle b | d_t a \rangle - \langle c | d_t b \rangle \right) \cdot |c\rangle$

but $\hat{\partial}$ is a <u>co</u>boundary.

Morse-Bott Theory

Suppose the set of crit pts of $h:(M,g) \to \mathbb{R}$ is a submfd N w/ components N_i & on any pt of N_i, the Hessian is nondegenerate on orthogonal directions to N_i. The # of neg eigenvalues of the Hessian is some p_i, the Morse index of N_i. We obtain a p_i-rank vect bundle over N_i called $\Lambda(N_i)$.

Then $V(\phi) = t^2 |\nabla h|^2$ vanishes on the N_i but is large elsewhere. Then, the wavef^{ns} (states) vanish rapidly away from the N_i.

Pick an N_i, call it N_0.

Claim: For large t, the low energy spectrum of H_t acting on states localized near N_0 converge to the spectrum of Δ on N_0.

Let $M(N_0)$ be a tubular nghd of N_0; it can be regarded as the normal bundle.

Let $\tilde{d}$ denote the exterior derivative on N_0 which extends to $M(N_0)$.

we find that

$$H_t = \overbrace{\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d}}^{\tilde{\Delta}} + H'$$

acts in transverse directions to N_0

For large t, H' has a similar form to the $\overline{H}_t$ we saw before (makes sense as the transverse directions are nondegenerate & bear similarities to the Morse setting)

Fix $n \in N_0$; then H' can be restricted to the fiber over n, F_n.

$H'|_{F_n}$ has a single zero energy state — all other states have energy proportional to t.

Rmk: Compare this to the Thom class of a bundle E^n which can be viewed as the unique cohomology class in $H^n_{cv}(E)$ which restricts to the generator of $H^n_c(F)$ on each fiber F

Let $|\alpha(m;n)\rangle$ be the zero energy state of H' on the fiber F_n at pt m. It is a p-form ($p = \text{ind}\, N_0$).

Claim: Similar to Born-Oppenheimer approx in molecular physics, the deg of freedom transverse to N_0 are frozen in their ground state $|\alpha\rangle$ b/c of the large energy associated to any excitation.

$\therefore$ we may write a low energy state $|\psi\rangle$ of H_ϕ in the form $|\psi(n,m)\rangle = |\chi(n)\rangle \otimes |\alpha(m,n)\rangle$. (Künneth formula or Leray-Hirsch thm)

$$\underset{H^*(M(N_0))}{|\psi(n,m)\rangle} \qquad \underset{\text{"}H^*(N_0)\text{"}}{|\chi(n)\rangle} \qquad \underset{H^*(F_n)}{|\alpha(m,n)\rangle}$$

The caveat is that $|\chi\rangle \in H^*(N_0)$ if $\Lambda(N_0)$ is orientable. If not, then $|\chi\rangle$ is a section of the de Rham complex of N twisted w/ the orientation bundle of $\Lambda(N_0)$.

$|\alpha(M,n)\rangle$ is annihilated by H' { so for large t,

the eigenval problem $H_t \psi = \lambda \psi$, $\psi = \chi \otimes \alpha$ reduces to $\tilde{\Delta} \chi = \lambda \chi$ on N_0.

The 0-eigenstates χ are in 1-1 correspondence to generators of the (twisted) cohomology

The approx we're making is to ignore α's dependence on N_0.
This is valid to lowest order in $1/t$.

$\Rightarrow$ nonzero energy states in the approx have nonzero energy in actuality for large t

In fact, their energies equal (for large t), the nonzero eigenval of the Laplacian on N.

We obtain inequalities for Morse-Bott theory.

The contribution of N_0 to the Morse polynomial is

$$t^{p} \bar{P}_t(N_0)$$

ordinary Poincaré poly or twisted Poincaré

$$\hat{P}_t(N_0) = \sum_k b_k(N_0)\, t^k$$

3. Killing Vector Fields

Let (M, g) be a cpt Riem mfd

def. A Killing vector field K on M satisfies

$$\mathcal{L}_K g = 0.$$

It may be viewed as an infinitesimal generator of an isometry of M i.e. its flow generates a 1-param family of isometries.

Fact: If (M, g) is cpt, K – Killing vect field, η – harmonic form, then $\mathcal{L}_K \eta = 0$

We fix such a K. Let $N = \{$vanishing pts of $K\}$

Regard K as an operator on forms by interior mult.

$$\iota_K$$

Then, let $s \in \mathbb{R}$ be fixed &

$$d_s \doteq d + s\,\iota_K$$

Note that d_s maps a p-form to a combination of a $(p+1)$ & $(p-1)$ form.

Let $V_+ = \Lambda^{even} T^*M$, $V_- = \Lambda^{odd} T^*M$.

So $d_s : V_\pm \longrightarrow V_\mp$.

Observe: $$d_s^2 = \underbrace{d^2}_{0} + s\,d\iota_K + s\,\iota_K d + s^2 \overset{0}{\cancel{\iota_K \iota_K}}$$

$$= s\,\mathcal{L}_K \quad \text{(Cartan's magic formula)}$$

Also, if d_s^* is the adjoint, using the fact that K is a Killing vect field, we can show that

$$-d_s^* = s\,\mathcal{L}_K$$

Let $H_s = d_s d_s^* + d_s^* d_s$ be our "Hamiltonian."

Main Theorem: The # of zero eigenvalues (multiplicity) of H_s is independent of s (for $s \neq 0$) & indep of any K-invariant Riem metric on M.

$$\text{The \# of zero eigenval of } \underset{(s\neq 0)}{H_s} = \sum_k b_k(N)$$

Betti #s.

Moreover, we know that when $s=0$, $H_s = \Delta$ - Laplacian on M.

Then the Hodge thm says:

$$\text{\# zero eigenval of } \Delta = \sum_k b_k(M).$$

The eigenval of H_s are smooth f^ns of s since the s-dependent terms are bounded operators. Then, for very small s, the # of 0 eigenval is no bigger than for $s=0$.

$$\Rightarrow \sum_k b_k(N) \leq \sum_k b_k(M)$$

This is not true in general, of course. N is specifically the fixed pts of flow generated by K.

In determining the # of 0-eigenval of H_s, for $s \gg 0$, we can express the Hirzebruch signature of M in terms of N.

We also obtain a version of the Lefschetz Fixed point thm. where the contribution of each component of N is an integer (its signature).

Lastly, dropping the condition that K is a Killing vect field, we can obtain from the $s \to \infty$ limit of H_s a proof of the Poincaré-Hopf thm.

These are all variants of the proofs based on the index theorem.

Let's return to our main goal: Count the zero eigenvalues of $H_s = d_s d_s^* + d_s^* d_s$ $(s \neq 0)$

Note: If $H_s \psi = 0$, then $0 = \langle H_s \psi, \psi \rangle$

$$= \langle d_s d_s^* \psi, \psi \rangle + \langle d_s^* d_s \psi, \psi \rangle$$

$$= \underbrace{|d_s^* \psi|^2}_{\geq 0} + \underbrace{|d_s \psi|^2}_{\geq 0}$$

$\Rightarrow d_s \psi = d_s^* \psi = 0.$

So $H_s \psi = 0$ iff $d_s \psi = d_s^* \psi = 0$.

Hence, if $\psi \in \operatorname{Ker} H_s$, then $\psi \in \operatorname{Ker} d_s^2 = \operatorname{Ker} \mathcal{L}_K$.

So ψ is invariant under the isometries generated by K.

So we restrict our attention to $\overline{V} \doteq \operatorname{Ker} \mathcal{L}_K$.

Since $d_s^2 = 0$ in $\overline{V}$, view d_s like a coboundary operator.

Using similar techniques as Hodge theory, one finds the # zero eigenvalues of $H_s = \dim\left(\ker d_s / \operatorname{Im} d_s\right)$.

The definition of d_s can be made independent of a metric since it relies only on the vect field K. So it is indep of K-invariant Riem metrics.

To show the # of 0-eigenvalues is indep of s (so long $s \neq 0$), we conjugate by $e^{\lambda P}$ (I don't think P is the momentum operator).

This does not change the dim of $\ker d_s / \operatorname{Im} d_s$.

So $e^{-\lambda P} d_s e^{\lambda P} = e^{-\lambda} d_{s'}$, $s' = s e^{2\lambda}$. Tuning λ, we see have our s-independence when $s \neq 0$.

These arguments work also on counting the # of even or odd 0-energy states. Let these be denoted n_+ & n_- for H_s acting on V_+ & V_-. Then n_+ & n_- are indep of s & $n_+ - n_- = \chi(M)$ ~ Euler characteristic

Next Goal: Let N_+ = sum of even Betti #s of N.

N_- = ~~#~~ odd ~~#~~.

Prove $n_{\pm} \geq N_{\pm}$.

Claim: we only need to show one of these since

$n_+ - n_- = N_+ - N_- = \chi(M)$. (b/c $n_+ - n_-$ is indep of s)

Assuming this formula,

$$n_+ \geq N_+ \implies N_- + \underset{\geq}{(n_+ - N_+)} = n_- \implies N_- \leq n_-.$$

Depending on whether $n \doteq \dim M$ is odd/even, we focus on showing $n_+ \geq N_+$ or $n_- \geq N_-$.

Let N_0 be any conn component of N & φ a diff form on N_0 which is closed but not exact.

Let $M(N_0)$ be a tubular nbhd of N_0; it has vect bundle structure.

$$\begin{array}{c} M(N_0) \\ \pi \downarrow \\ N_0 \end{array}$$

Let $\hat{\varphi} = \pi^* \varphi$. The action of K on $\hat{\varphi}$ is to lift K to $M(N_0)$ then use interior product:

lift K to fibers to get a vect field on $M(N_0)$.

Then,

$$\begin{array}{ccc} H^* M(N_0) & \xrightarrow{\iota_K} & H^{*-1} M(N_0) \\ \uparrow \pi^* & \circlearrowleft & \uparrow \pi^* \\ H^* N_0 & \xrightarrow{\iota_K} & H^{*-1} N_0 \end{array}$$

$\therefore \ \iota_K \hat{\varphi} = 0$.

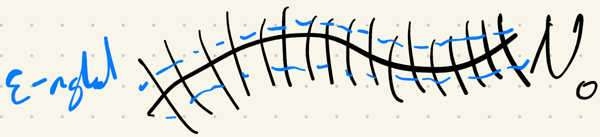
Also, $d\hat{\varphi} = \pi^* d\varphi = 0 \Rightarrow d_S \hat{\varphi} = 0$.

Also, on $M(N_0)$, it is impossible to satisfy $\hat{\varphi} = d_S \alpha$. This is b/c, on N_0, $K \equiv 0$ so $d_S = d$ on N. Then $\hat{\varphi} = d_S \alpha \Rightarrow \varphi = d\alpha$ but φ is not exact.

However, on $\partial M(N_0)$, $d\hat{\varphi}$ & $d_S \hat{\varphi}$ are nonzero.

We modify them as follows:

Let $K^2 = g(K, K)$; it vanishes only on N. Let M_ε be pts of M s.t. $K^2 \leq \varepsilon$ for some $\varepsilon > 0$. Let ε be small so that $M_\varepsilon \subset M(N_0)$.



Let $\phi : M(N_0) \to \mathbb{R}$ be s.t. $\phi|_{N_0} \equiv 1$, $\phi(x) = 0$ for $K^2(x) \geq \varepsilon$

Let $\hat{K} = g$-dual of K. B/c K is a Killing v.f.

One can show $\iota_K d\hat{K} = -d(K^2)$ (I can't show it)

Define $\sigma = \phi(K^2) + \frac{1}{s}\phi'(K^2)\, d\hat{K} + \frac{1}{2s^2}\phi''(K^2)\, d\hat{K} \wedge d\hat{K}$

$+ \frac{1}{3s^3}\phi'''(K^2)\, d\hat{K} \wedge d\hat{K} \wedge d\hat{K} + \cdots$

The series terminates since $n = \dim M < \infty$.

Claim: If n even, $d_s \sigma = 0$. If n odd, $d_s \sigma = 0$ except in deg n.

Say $n = 2$. Then $\sigma = \phi(K^2) + \frac{1}{s}\phi'(K^2)\, d\hat{K}$.

$$d\sigma = \phi'(K^2)\underbrace{d(K^2)}_{= -\iota_K d\hat{K}} + \underbrace{\frac{1}{s}\phi''(K^2)\underbrace{d(K^2)}_{= -\iota_K d\hat{K}} \wedge d\hat{K}}_{\text{deg } 3 \text{ so } = 0 \text{ as } n = 2} + \overbrace{\frac{1}{s}\phi'(K^2)\, d^2\hat{K}}^{0}$$

$= -\phi'(K^2)\, \iota_K d\hat{K}$ ← cancels

Also, $s\,\iota_K \sigma = s\,\underbrace{\iota_K \phi(K^2)}_{\substack{= 0 \\ \text{b/c deg} = -1}} + \frac{s}{s}\phi'(K^2)\, \iota_K d\hat{K}$. $\therefore d_s \sigma = 0$.

Based on these patterns, the claim is confirmed.

Let $\chi \doteq \hat{\varphi} \wedge \sigma$. Assume φ is even(odd) if n is even(odd).

Then $d_s \chi = (d + s\iota_K)(\hat{\varphi} \wedge \sigma)$

$$= \underbrace{d\hat{\varphi}}_{=0} \wedge \sigma \pm \hat{\varphi} \wedge d\sigma + \overbrace{s\iota_K \hat{\varphi}}^{=0} \wedge \sigma$$

$$= \pm \hat{\varphi} \wedge d\sigma.$$ [I think this is correct. However, Witten says $d_s \chi = 0$.]

Also, χ is <u>not</u> d_s-exact. If it were, that implies φ is exact which it is not.

So for every even (or odd) cohom class of N, we produced an object χ which is closed but not exact in the sense of d_s.

I think if $[\chi] = [\chi']$, then $[\varphi] = [\varphi']$.

Then, depending on n even/odd, we've shown $n_+ \geq N_+$ or $n_- \geq N_-$.

Now to prove converse inequalities: $N_+ \geq n_+$ & $N_- \geq n_-$.

We compute: $H_s = d_s d_s^* + d_s^* d_s$, let $\hat{K} = {}^g\text{dual of } K$

$$= (d + s\iota_K)(d^* + s\underbrace{\hat{K}\wedge}_{\text{wedge operator}}) + (d^* + s\hat{K}\wedge)(d + s\iota_K)$$

$$= dd^* + d^*d + s\,d(\hat{K}\wedge) + s\iota_K d^* + s^2 \iota_K \hat{K}\wedge \quad = g(K,K) = K^2$$

$$+ s\,d^*\iota_K + s\hat{K}\wedge d + s^2 \hat{K}\wedge\iota_K.$$

cancels

$$= s(d\hat{K}\wedge - \hat{K}\wedge d)$$

not sure what to do w/ these terms

$$= \Delta + s^2K^2 + s(d\hat{K})\wedge + s(\iota_K d^* + d^*\iota_K) + s^2\hat{K}\wedge\iota_K$$

Witten says we get

$$H_s = \Delta + s^2K^2 + s\left((d\hat{K})\wedge + c(d\hat{K})\right)$$

adjoint of $(d\hat{K})\wedge$.

The potential energy is $V(\phi) = s^2 K^2$ (cf. Morse situation w/ $s^2 |df|^2$)

The proof is similar to the Morse case

Assume K has only isolated zeros. By Poincaré-Hopf, if the indices add up to nonzero, then $\chi(M) \neq 0 \Rightarrow \dim M = n$ is even.

Claim: When K has only isolated zeros, $N_- = 0$;

$N_+ = \#$ of zeros of K.

Pf: Near any zero A of K we can find local coord centered at A for K & H_s can be approx. by a $\overline{H}_s$. Similar to the Morse setting, one can diagonalize $\overline{H}_s$ & $\exists!$ zero eigenval, all others are on the order of s.

The one zero eigenval is in V_+. So there are N_+ states in V_+ whose energy does not diverge w/ s & none in V_-.

So $n_+ \leq N_+$, $n_- = N_- = 0$. By prev inequality,

$$n_+ = N_+.$$

If K has nonisolated zeros, the discussion is like the Morse-Bott setting.

Now, $H_s = (d_s + d_s^*)^2$ & $d_s^2 + (d_s^*)^2 = 0$. So

$$H_s = d_s d_s^* + d_s^* d_s. \text{ Let } D_s = d_s + d_s^*$$

So $\{\text{zeros of } H_s\} \xleftrightarrow[\text{cor}]{1\text{-}1} \{\text{zeros of } D_s\}$

Decompose $V = V_+ + V_-$ & $D_s = D_{s+} + D_{s-}$.

Note: $D_{s\pm} : V_\pm \longrightarrow V_\mp$.

It can be shown that $\mathrm{Ind}(D_{s+})$ is indep of s

& so when $s=0$, $\mathrm{Ind}(D_{s+}) = \chi(M)$.

When s large, $\mathrm{Ind}(D_{s+}) = N_+ - N_- = \chi(N)$.

So $\chi(M) = \chi(N)$, N is the zero set of K