Light Rays & Black Holes II

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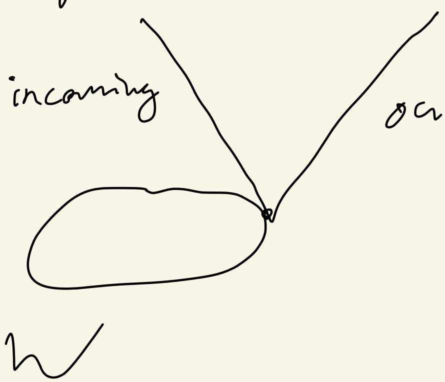
Let W be a codim 2 spacelike surface. It has two families of future-going null geodesics orthogonal to W.



Pick the outgoing family. It is a $(d-1)$ mfd U on which the metric is degenerate ("null")

Null geodesics don't have proper time but $\exists$ affine parameter u [coord, & this is well defined up to $u \mapsto au+b,\ a,b\in\mathbb{R}$)

which makes the geodesic eqn simpler:

$$\frac{d^2x^\lambda}{du^2}=0.$$

Can make u zero along W & then the metric can be written in these coord: (x^i, u)

The metric of U is $ds^2 = g_{ij}(\vec{x}, u)\,dx^i dx^j$.

Note the degeneracy b/c du does not appear.

The null Raychaudhuri eqn is the Einstein eqn

$$R_{uu} = 8\pi G T_{uu}.$$

Let $A = \sqrt{\det g}$, $\dot{A} = \partial_u A$, $\theta \doteq \frac{\dot{A}}{A}$.

Equiv, $\theta = \frac{1}{2}\operatorname{tr}(g^{-1}\dot{g})$. Let M_j^i = trace free part of $g^{-1}\dot{g}$ (the shear)

Null Energy Condition: At each pt ξ in each local Lorentz frame, $T_{uu} \geq 0$. This is not affected by a cosmological constant & is satisfied by any of the usual relativistic classical fields.

The strong energy condition <u>is</u> impacted by cosmo. const.

Assuming NEC, the Einstein-Raychaudhuri eqn says that

$$\partial_u \left(\frac{\dot{A}}{A}\right) + \frac{1}{d-2}\left(\frac{\dot{A}}{A}\right)^2 \leq 0.$$

As in the timelike case, if at a pt W, the initial value of the null expansion is $\dot{A}/A = -\lambda$, $\lambda > 0$, then the geodesic will reach a focal pt $A = 0$ (or a singularity $\dot{A} = \infty$) at a value of the ~~at~~ the parameter

$$u \leq \frac{d-2}{\lambda}.$$

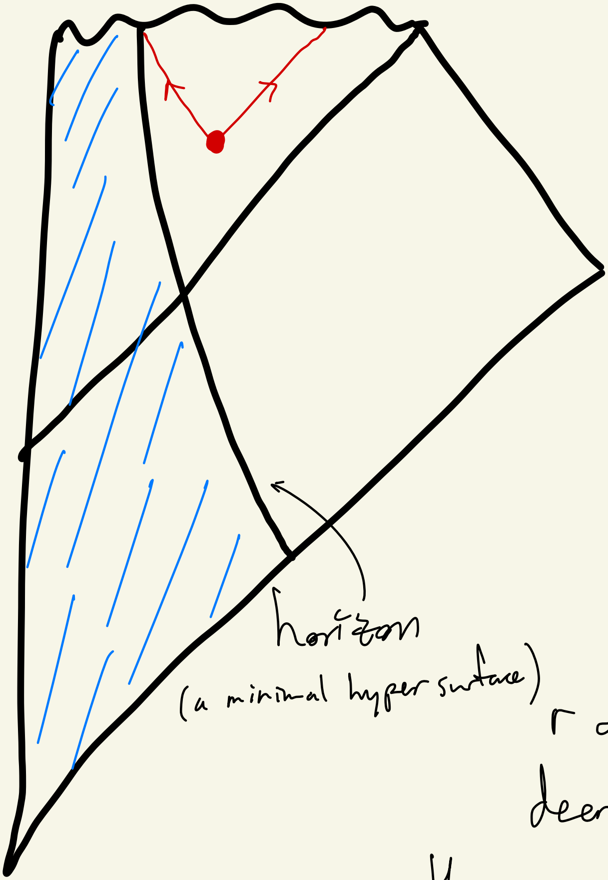
$\therefore$ like the timelike case, knowledge of focal pts in the null case gives singularity thms.

For each W, there are two families of outgoing ortho null geodesics. incoming & outgoing. So there are two null expansions which can be positive or negative

def (Penrose): A trapped surface is a compact, spacelike codim 2 mfd W s.t. both null expansions are negative.

Motivating example: surface beyond horizon of a Schwarzschild black hole

in 4-dim



blue region = collapsing star

wiggly black line = singularity

red dot = 2-sphere behind horizon

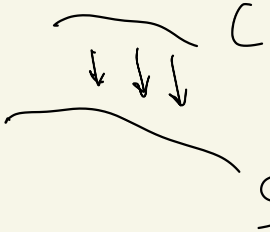
it has area $4\pi r^2$.

As we travel along the future-going causal curves, r <u>decreases</u> so the area decreases as well. Hence, the name <u>trapped surface</u>.

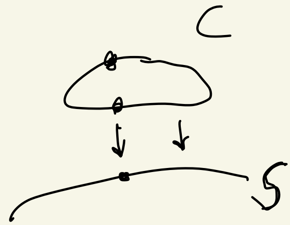
In more detail, since $\dot{r} < 0$, $r > 0$,

then $\frac{\dot{r}}{r} < 0 \Rightarrow \frac{\dot{A}}{A} < 0 \Rightarrow$ both expansions are negative.

In a globally hyperbolic spacetime M w/ initial value surface S, if C is any achronal set, it is topologically equivalent to a subset of S. Just flow S in the time direction.



if C is not achronal, then this projection is not injective.



Observe: Let (M^D, S) be a globally hyperbolic spacetime w/ S being a <u>noncompact</u> initial val surf. Then for any subset C, $\partial J^+(C)$ cannot be cpt b/c $\partial J^+(C)$ is an achronal mfd w/ $\dim = \dim S = D-1$. So it will be top equiv to a submfd of S.

$\boxed{\text{But noncpt, conn mfd } S \text{ cannot have } \underline{\text{cpt}}\ \underline{\text{codim } 0} \text{ submfds.}}$

Now, in a spherically symmetric case, one can solve the eqn & demonstrate the formation of a singularity, e.g. asymptotically flat space

Use the fact: outside the star, the Schwarzschild solution is unique. In vacuum, a spherically symmetric solution is unique & its collapse, leads to the same outcome, as w/ Schwarzschild

Penrose's motivating question: Does infalling matter still collapse to a singularity if we do not assume spherical symmetry?

He introduced trapped surfaces & show singularities occur once a trapped surface forms. (at least predictivity).

Trapped surfaces are stable under small perturbations of the metric so these singularities form generically.

Precise thm: Let (M, g) be globally hyperbolic & noncpt. Suppose (Penrose) M contains a cpt trapped surface C. Then M is geodesically incomplete: at least one ortho null geodesic from C cannot be continued indefinitely into the future.

Caveat: the reason the geodesic cannot be continued is not b/c it ends in a singularity. There are examples where the geodesic cannot be continued yet there is no singularity.

Counter Examples supplied by Anti de Sitter space.

Singularity means you can't continue spacetime as a smooth mfd.

pf: Suppose every future going null geodesic is extendable indefinitely. Since C is a cpt trapped surface, its null expansions $\mathring{A}/A < -\lambda, \lambda > 0$ is everywhere negative ; so there exists this bound $-\lambda$.

The future-going null geodesics are extendable indef so they can go beyond affine distance $\frac{D-2}{\lambda}$. By Ray. eqn, they can extend beyond their 1st focal pt.

Now $\partial J^+(C)$ consists of pts on the future-going ortho null geodesics from C that are not beyond focal pts. If l is such a geodesic which does extend beyond its 1st focal pt, then the part of l in $\partial J^+(C)$ is cpt. C cpt $\Rightarrow \partial J^+(C)$ is cpt b/c of the cpt segments.

However, we've proven before that $\partial J^+(C)$ cannot be cpt. ↯

$\therefore$ not every future-going ortho null geodesic is extendable indefinitely.

This then doesn't tell us much about the region inside a black hole. However, the ideas developed to lead quite easily into understanding black holes so long as we make one more assumption.

we assume nothing worse happens: i.e. no formation of a naked singularity visible to an outside observer

Why is this worse? It seems we just don't know how to deal w/ them. But if there are no naked singularities, we can develop a theory.

Penrose introduced "Cosmic censorship": in any localized process in an asymptotically flat spacetime (e.g. gravitational collapse), the region in the far distance & far future continue to exist.

Moreover, there is no naked singularity visible to a distant observer.

So singularities are to be hidden behind a horizon.

Witten: if cosmic censorship is true, that would be rather surprising. This is b/c the classical Einstein eqns have no obvious stability properties.

Some reasons to believe in cosmic censorship: The black hole collisions observed by LIGO did not generate naked singularities (nor have simulations). They just form larger black holes.

This question about whether cosmic censorship is true, is perhaps the outstanding unanswered question about classical general relativity.

Let's assume cosmic censorship.

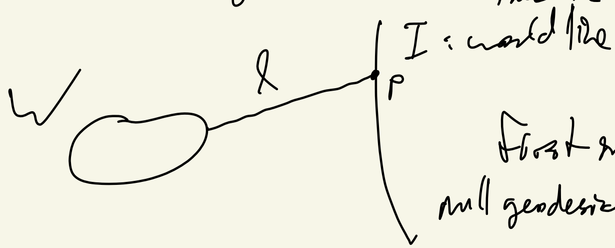
def. A black hole region B in spacetime is the region not visible to an outside observer.

More precisely: Let I be the worldline of a timelike observer who is more or less at rest over great distances, in the asymptotically flat region observing whatever happens.

Let $J^-(I)$ = causal past; all pts from which the observer could receive a signal. Then $B = M \setminus J^-(I)$ & the horizon $H = \partial B$.

Prop: A trapped surface W is in B. So signals cannot escape.

pf: Suppose a signal ℓ escapes. Then there is a first instance it is observed by the observer.



This signal is prompt as it is the first instance. ∴ it is a future going null geodesic, $\perp$ to W, no focal pts

Since W is trapped, there is a focal pt on ℓ within a known bdd affine distance from W. The observer can be placed far away, however, beyond all focal pts.

This is a contradiction. ↯ $\blacksquare$

Recap: 1. Penrose showed incompleteness when there is a trapped surface.

2. Trapped surfaces are stable under small perturbations of the geometry

Need cosmic censorship assumption

3. trapped surfaces have to be inside a black hole region.

4. There are trapped surfaces for the explicit Schwarzschild & Kerr solutions

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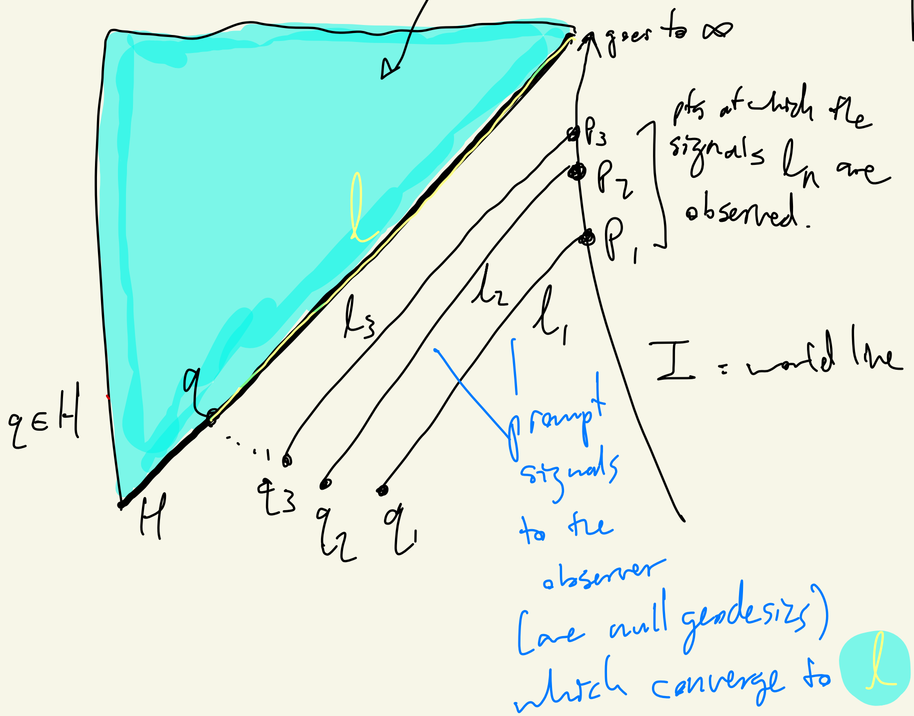
∴ For geometries close to Schwarzschild / Kerr, black holes exist generically. (Again assuming cosmic censorship)

Also, if $W \subset B$, then $J^+(W) \subset B$

Note: Black holes can merge but cannot split in the future.

Horizon Generators

Later, we'll prove H is light like.



$q_n \to q$

$l_n \to l$.

However, as $n \to \infty$, the p_n are farther & farther in the future, i.e. writing $l_\infty \doteq l$, p_∞ is not on I, it is at time $= +\infty$

So the signal l never arrives to be observed.

Claim: l is completely contained in H.

It clearly can't be in the interior of B but it can't be exterior to it either, b/c then there would be communication to the observer.

In the same situation as the prev. page:

Now pick S, an initial value surface & $q \in W = S \cap H$.

It must be the case that $\ell \perp W$, & it cannot have focal pts. i.e. ℓ is a prompt causal path from W.

The ortho null geodesics from W that stay in the horizon are called horizon generators. Every pt in W is contained in a horizon generator. Together, they sweep a $(D-1)$ mfd H'.

$H' \subset H$ & near W, $H' = H$.

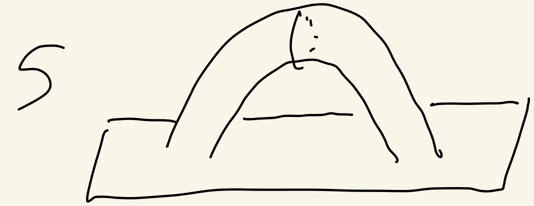
Hawking Area Thm: The area of a black hole horizon can only increase as time passes; i.e. area msred on initial val surface S' to the future of S is at least as large as the area msred at S.

(again, we still assume cosmic censorship)

Additional Results to discuss:

- Topological Censorship
- Gao-Wald thm
- Average null energy condition (ANEC)

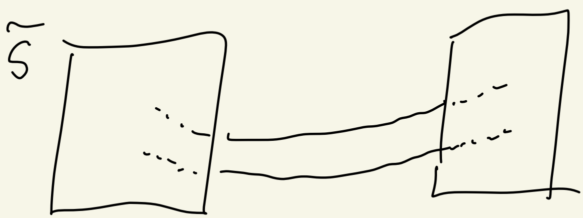
Topological Censorship: let M be asymptotically flat & S an initial value surface. Although there may be a wormhole in S, a causal path cannot go through the wormhole & come out the other side



In 3+1 spacetime, $\dim S = 3$.

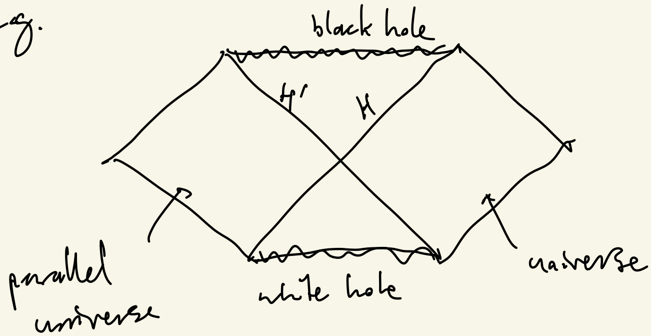
Existence of a wormhole $\Rightarrow \pi_1(S) \neq 0$.

Replace S w/ its universal cover $\tilde{S}$. We get an equiv picture in which $\tilde{S}$ is simply conn but has more than one asym flat region.



Motivating: The analytically continued Schwarzschild solution:

e.g.



Topological censorship holds in general, not just for Schwarzschild

Topological Censorship actually holds under a weaker condition: Average Null Energy Condition (ANEC).

ANEC: Let l be a null geodesic w/ affine parameter u running to ∞ at both ends. Then

$$\int_{-\infty}^{\infty} T_{uu}\, du \geq 0$$

in the sense that the operator has nonnegative expectation value in every quantum state.

Caveat: ANEC is <u>not</u> universally true in QFT. It is believed to be true only for null geodesics that are achronal & complete in both directions.

We diverge & discuss the Gao-Wald thm before returning to ANEC.

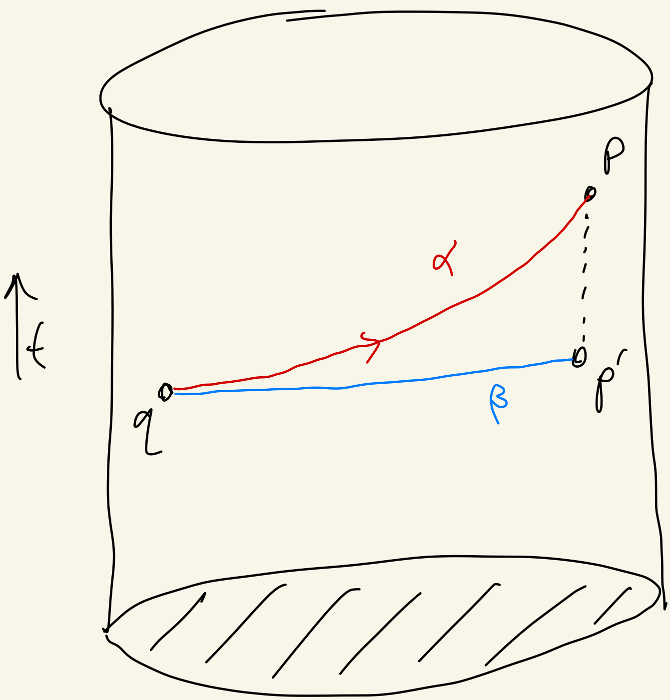
Gao-Wald Thm: The AdS/CFT correspondence is compatible w/ causality (under ANEC hypothesis is enough)

More precisely: Let M be any AdS spacetime. By adding some pts at spatial infinity, we get a partial conformal compactification of M. The pts at ∞ make a Lorenz sig mfd N. AdS/CFT duality says that a gravitational theory on M is equiv to some conformal field theory on N.

To discuss causality. Let $q, p \in N$ w/ causal path b/w them, contained in N. Say the causal path α is a prompt null geodesic on the boundary N.

Is there a pt $p' \in N$ to the past of p & a causal path β from q to p', contained in M? If so, then the duality is in conflict w/ causality.

e.g. Let $M = T^*D^2$, $N = T^*S^1$.



α is confined to N

β passes through M

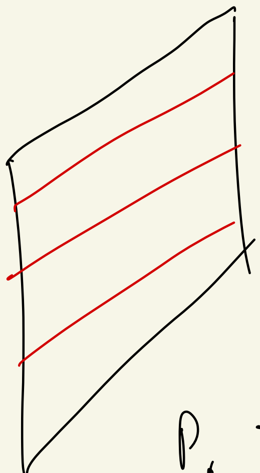
ANEC in 2-dim Minkowski spacetime is just positivity of energy.

Let $x^\pm = x \pm t$ be light cone coord. A null geodesic is for example, $x^- = 0$. The null energy density T_{++} integrated on the null geodesic is $P_+ = \int_{x^-=0} T_{++}\, dx^+ \geq 0$.

It vanishes only for the vacuum state in any Lorenz-invariant QFT.

For $D > 2$, there's more interesting behavior.

Let $H = \{x^- = 0\}$ be a null hypersurface.



H is ruled by null lines (geodesics complete in both directions)

$$P_+ = \int_H dx^+ d\vec{x}\, T_{++}$$

$A(\vec{x}) = \int dx^+ T_{xx}$. So $P_+ = \int A(\vec{x})\, d\vec{x}$.

ANEC says $A(\vec{x}) \geq 0\ \forall \vec{x}$ & any x^-.

Now, $[P_t, A(\vec{x})] = 0$. Let Ω = unique vacuum state.

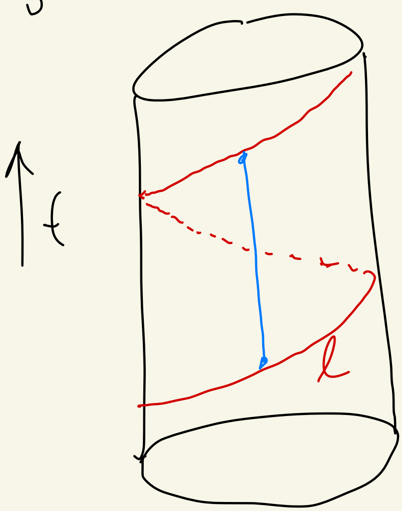
$\therefore A(\vec{x})\Omega$ & Ω have the same P_t $\Rightarrow A(\vec{x})\Omega$ is a multiple of Ω. In fact, $A(\vec{x})\Omega = 0$.

Not sure what we should take away...

Counterexample to ANEC

$M = T \times S^1$

$ds^2 = -dt^2 + dx^2, \; x \in [0, 2\pi R]$



The ANEC integral of l is negative. But it is not achronal. One can find a timelike geodesic from l to itself easily; e.g. the blue line.