Examples from Complex Geometry

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1 Complex Analysis

Example 1.1. Two Heuristic "Proofs" of the Fundamental Theorem of Algebra: Let $p(z)$ be a polynomial of degree $n > 0$; we can even assume it is a monomial. We also know that the number of zeros is at most n. We show that there are exactly n.

1. Proof 1: Recall that polynomials are entire functions and that Liouville's Theorem says that a bounded entire function is in fact constant. Suppose that p has no roots. Then $1/p$ is an entire function and it is bounded. Thus, it is constant which means p is a constant polynomial and has degree 0. This contradicts the fact that p has positive degree. Thus, p must have a root α_1. We can then factor out $(z - \alpha_1)$ from $p(z) = (z - \alpha_1)q(z)$ where $q(z)$ is an $(n-1)$-degree polynomial. We may repeat the above process until we have a full factorization of p.

2. Proof 2: On the real line, an algorithm for finding a root of a continuous function is to look for when the function changes signs. How do we generalize this to $\mathbb{C}$? Instead of having two directions, we have a whole S^1 worth of directions. If we use colors to depict direction and brightness to depict magnitude, we can plot a graph of a continuous function $f : \mathbb{C} \to \mathbb{C}$. Near a zero, we'll see all colors represented. If we travel in a loop around any point, we can keep track of whether it passes through all the colors; around a zero, we'll pass through all the colors, possibly many times. Thus, the winding number of a zero is a nonzero integer.

   For z with large norm, $p(z) \approx z^n$. We know that the winding number of the loop $\gamma(t) = Re^{it}$ (for any $R > 0$) around 0 is n. Moreover, suppose we have two bounded regions with piecewise smooth boundary and they share a nonempty piece of boundary. Then the winding numbers of the two regions add up to the winding number of the boundary of the union of the two regions.

   For example, if we consider a large rectangle that has been cut into two rectangles, the two rectangles share a side. However, when we compute the winding numbers of each smaller rectangle and add them together, we find the shared side contributes some positive amount from one rectangle and the negative amount from the other rectangle, thereby canceling any contribution. What remains then is the winding number of the large rectangle. This is a special instance of phenomena which appears in Stokes Theorem.

   This means for a large enough loop γ, it's winding number is n and when we split the disk that γ bounds into two pieces X and Y. Suppose the two pieces have boundary loops with winding numbers x, y such that $x + y = n$. Suppose y is zero. Then the region Y has no zeros and we can focus on X and further subdivide it. In this fashion, we may iterate and converge towards the zeros.

Since n is an integer and winding numbers are integers, we can eventually find n small regions all with winding number 1. There is no fear of some of the winding numbers being negative as polynomials are orientation preserving maps.

2 Examples of Complex Manifolds/Varieties

Example 2.1. Consider the two curves in $\mathbb{C}P^2$ cut out by the following equations:

$$C_1 := \{x^2 + y^2 + z^2 = 0\}$$
$$C_2 := \{xy + yz + xz = 0\}$$

The two curves have genus 0 by the genus-degree formula and are smooth since the partial derivatives only vanish at $(0, 0, 0)$ which is not in $\mathbb{C}P^2$. So they're embedded spheres. By Bézout's theorem, we expect four intersection points.

I'm sure there is a general principle for finding solutions. But as a guess, we see that the 3rd roots of unity sum up to 0. Moreover, the squares of the 3rd roots of unity are again, the 3rd roots of unity. Thus, let $x = \omega := e^{2\pi i/3}$, $y = \omega^2$, $z = \omega^3 = 1$. It's easy to see that $[x : y : z]$ satisfy both equations. We get one additional solution by permuting: $[x : z : y]$ but no more because in $\mathbb{C}P^2$, $[x : y : z] = \lambda[x : y : z]$ for any $\lambda \in \mathbb{C}^*$.

The other two solutions come from the 6th roots of unity. Let $x = \eta := e^{\pi i/3}$, $y = \eta^3$, $z = \eta^5$. Then, $x^2+y^2+z^2 = \eta^2+\eta^6+\eta^{10} = \omega+\omega^2+\omega^3 = 0$. And $xy+yz+xz = \eta^4+\eta^8+\eta^6 = \omega + \omega^2 + \omega^4 = 0$. As before, we get two solutions because we can get an additional solution by permuting.

Example 2.2. There's a common algebraic geometric way of thinking of blowing up $\mathbb{C}P^2$ at a point; you add in some $\mathbb{P}^1$ and you get a bundle where the total space is contained in $\mathbb{C}^2 \times \mathbb{C}P^2$. We're just introducing all the directions (lines) one can move in through the point. Topologically, you can think of a blowing up as connect summing: $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$.

A third way to view the blow up, which I learned from Martin Roček, is to consider the Fubini-Study form. We can consider, in $M = \mathbb{C}^3 - \{0\}$, the function $\log(|z_1|^2 + |z_2|^2 + |z_3|^2)$ and apply $\partial\bar{\partial}$ to it. However, let's choose $\alpha, \beta \in \mathbb{C}^3 - \{0\}$ and consider now $\log(|z_1|^2 + |z_2|^2 + |z_3|^2) + \log(|\alpha \cdot z_1|^2 + |\beta \cdot z_2|^2)$. For example, if $\alpha = (1, 0, 0), \beta = (0, 1, 0)$, then we're just adding $\log(|z_1|^2 + |z_2|^2)$. Applying $\partial\bar{\partial}$ and letting it descend to $\mathbb{C}P^2 = M/\mathbb{C}^*$ gives us a metric which behaves singularly at a point defined by α and β (it's like a pair of lines crossing). Blowing up at that point resolves the issue. We need a pair α, β since $\mathbb{P}^1$ is defined from $\mathbb{C}^2 - \{0\}$. The form we get might not be Kähler as it might be negative; it depends on the α and β but generically, it should be Kähler.

If we wish to blow-up at more points, take

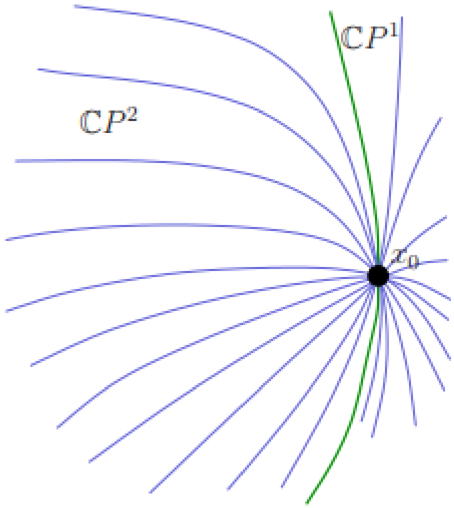
$$\log(|z_1|^2 + |z_2|^2 + |z_3|^2) + \sum_{j=1}^{\ell} \log |\alpha_j \cdot z_1|^2 + |\beta_j \cdot z_2|^2.$$

Here, $\ell = 0, ..., 8$; if $\ell = 9$, then we have strange behaviors. I think the result is no longer Fano and the underlying topological manifold can admit infinitely many smooth structures. 9 may be the magic number because the dimension of cubic curves in $\mathbb{C}P^2$ is 9. This corresponds, I think, to looking at polynomials $p(x, y, z) = \alpha x^3 + \beta y^3 + \gamma z^3 + \delta x^2 y + \epsilon x^2 z + \zeta y^2 z + \eta y^2 z + \theta z^2 x + \iota z^2 y + \kappa xyz$. That's 10 variables but then we projectivize or something to get 9. If someone knows why 9 is so special, please tell me.

Example 2.3. Consider a hyperplane through the origin in $\mathbb{C}^{n+1}$. Then under the quotient map $q : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$, $q(H)$ is homeomorphic to $\mathbb{C}P^{n-1}$ and the complement of this image is $\mathbb{C}^n$. So an alternative construction of $\mathbb{C}P^n$ comes recursively: attach a $2n$-cell to $\mathbb{C}P^{n-1}$.

Let's take $\mathbb{C}P^2$ for concreteness. For any $\zeta \in \mathbb{C}$, we claim that the holomorphic embedding $u_\zeta : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (z, \zeta)$ extends naturally to a holomorphic embedding of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. Indeed, using ι_2 to include $\mathbb{C}^2$ in $\mathbb{C}P^2$, $u_\zeta(z)$ becomes the point $[z : \zeta : 1] = [1 : \zeta/z : 1/z]$, and as $z \to \infty$, this converges to the point $x_0 := [1 : 0 : 0]$ in the sphere at infinity.

One can check using alternate charts that this extension is indeed a holomorphic map. The collection of all these embeddings $u_\zeta : \mathbb{C}P^1 \to \mathbb{C}P^2$ thus gives a very nice decomposition of $\mathbb{C}P^2$: together with the sphere at infinity, they foliate the region $\mathbb{C}P^2 - \{x_0\}$, but all intersect precisely at x_0. Therefore, we have that $\mathbb{C}P^2$ is a sphere at infinity ($\mathbb{C}P^1$) with $\mathbb{C}^2$ attached.



NB: This example is from Chris Wendl's *Lectures on Holomorphic Curves in Symplectic and Contact Geometry.*

Example 2.4. Let us consider curves in $\mathbb{C}P^2$. These are codimension 1 subvarieties and thus, they are cut out by single homogeneous polynomials of some degree d. The genus-degree formula gives $g = \frac{1}{2}(d-1)(d-2)$.

Here is a heuristic argument by Qiaochu Yuan for this formula. First consider the singular curve of degree d given by d lines in general position, so that every pair of lines intersects exactly once but otherwise there are no intersections. Topologically this gives a collection of d spheres each pairwise intersecting in a point. If we perturb the coefficients of the singular curve, it will become smooth; topologically the d spheres become pairwise connected by tubes. After using $d-1$ of these tubes to connect the spheres in a line, to obtain a sphere, the remaining $\binom{d}{2} - (d-1) = \frac{1}{2}(d-1)(d-2)$ tubes each increase the genus of the resulting surface by 1.

Example 2.5. The Thom space of the line bundle $\mathcal{O}(1) \to \mathbb{C}P^n$, which is the dual of the tautological line bundle of $\mathbb{C}P^n$, is in fact $\mathbb{C}P^{n+1}$. Locally, the line bundle appears as $\mathbb{C}P^n \times \mathbb{C}$ but locally on $\mathbb{C}P^n$, it appears as $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$. These are all e_{2n+2} cells and adding single point to glue them all together shouldn't really change the topology much. I think what we get it is now a single e_{2n+2} and at "infinity" or the zero section, we have a $\mathbb{C}P^n$. Thus, the Thom space of $\mathcal{O}(1) \to \mathbb{C}P^n$ is $\mathbb{C}P^{n+1}$.

This gives an embedding $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$; in the limit, we find that there's an embedding of $BU(1) = \mathbb{C}P^\infty \hookrightarrow MU(1)$; here $MU(1)$ is the Thom space of the dual of the universal bundle. But $MU(1)$ is also $\mathbb{C}P^\infty$! So we $\mathbb{C}P^\infty \hookrightarrow \mathbb{C}P^\infty$ and this is a homotopy equivalence. Being a homotopy equivalence is very important; it gives us a backwards map $f : MU(1) \to BU(1)$; since we can define universal Chern classes on $BU(1)$, we can now pull them back by f to $MU(1)$. Jiahao says this is very important in cobordism theory.

Example 2.6. Consider the K3 surface defined in $\mathbb{C}P^3$ by the homogeneous polynomial $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$. This one is usually denoted by $V^2(4)$ though all other K3's are diffeomorphic to it. This is a simply connected compact spin 4-manifold and by Rokhlin's theorem, its signature must be divisible by 16. Its signature is in fact, exactly 16. This has something to do with the fact that in the Hodge diamond, we see the 2nd row is 1, 20, 1 but the 20 breaks up into 19 and 1. Then the dimension of the maximum positive and negative spaces is given as $(19, 3)$; $16 = 19 - 3$.

Define an involution as follows $[z_0 : z_1 : z_2 : z_3] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2]$. Let $X = V^2(4)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ action is given by this involution. X is a orientable compact 4-manifold usually called an Enrique surface but $\sigma(X) = 8$ so X cannot be spin (by Rokhlin's theorem).

2.1 Some Examples from Huybrecht's *Complex Geometry*

Example 2.7. The Hopf manifolds are important examples of complex but not symplectic manifolds (and hence, non-Kähler and non-projective). The construction is to choose some $\lambda \in (0, 1) \cup (1, \infty)$ and let $\mathbb{Z}$ act on $\mathbb{C}^{n+1} - \{0\}$ by $k \cdot (z_1, ..., z_{n+1}) = (\lambda^k z_1, ..., \lambda^k z_{n+1})$. Then, the quotient $X = (\mathbb{C}^{n+1} - \{0\})/\mathbb{Z}$ is diffeomorphic to $S^1 \times S^{2n+1}$. Observe that when $n = 0$, then we get a torus and when $n = 1$, we have something diffeomorphic to $S^1 \times S^3$. Since S^3 is the total space of a Hopf fibration over S^2 with S^1 fibers, the Hopf surface is like an elliptic fibration over S^2 but isn't projective.

Also, it is well known by the Hodge Decomposition Theorem that the odd Betti numbers of a Kähler manifold should be even. In dim 2, this is sufficient. But in higher dimensions, we have non-Kähler manifolds that whose odd Betti numbers are even. An easy example is $X = S \times S$ where $S \cong S^1 \times S^3$ is a Hopf surface. Then $b_1 = 2, b_2 = 1, b_3 = 2, b_4 = 4$. This is an interesting example because the even Betti numbers are also positive which is one of the first conditions to check whether something is symplectic. So at first glance, X satisfies two necessary topological conditions for being Kähler yet is not Kähler.

An interesting question in general when given a smooth manifold which satisfies these two basic topological conditions is whether there actually is symplectic structure and whether it can be compatible with some deformation of the complex structure to obtain a Kähler structure. It is easy to see that $b_2(X)$ is generated from the 2-forms on the T^2 factor. When taking wedge products with itself, the cohomology will vanish; i.e. will be exact. So there is no symplectic form.

Still, it can be interesting to find a nondegenerate 2-form; i.e. an almost symplectic structure. A quick idea is the following. T^2 has a symplectic form and S^3 a contact form α which means $d\alpha$ is symplectic when we restrict to the distribution defined by $\ker \alpha$. The Reeb vector field is uniquely defined from this. So let's call R_1, R_2 the Reeb vector fields on the first and second copies of S^3. Then define a 2-form θ on $S^3 \times S^3$ which vanishes off of $D = \text{span}\{R_1, R_2\}$ and is nondegenerate on D. Let $\Omega = \omega + d\alpha_1 + d\alpha_2 + \theta$. I think this should work as an almost symplectic form (or some perturbation) since it should be nondegenerate. Again, to reiterate the above, this cannot be a symplectic form as it doesn't give a volume form; all the volume is concentrated on T^2.

There's an even easier approach. $T^2 \times S^3 \times S^3$ has trivial tangent bundle because it's a Lie group which are always parallelizable. Actually, we can see it's parallelizable just by the fact that all closed, oriented 3-manifolds are parallelizable. $\mathbb{R}^8$ has symplectic structure so let's just use it. If the complex structure is compatible with our almost symplectic form, is that called a nearly Kähler structure?

Also, the complex structure of X is such that the diagonal Hopf surface is a complex submanifold of X. If we have a complex submanifold in a Kähler submanifold, it is automatically Kähler. I wonder if this result holds when we put "nearly Kähler" in place of Kähler.

Example 2.8. Iwasawa 3-fold. Let G be the complex Lie group of upper triangular 3×3 matrices with 1's down the diagonal. It is a subgroup of $GL(3, \mathbb{C})$ and is biholomorphic to

$\mathbb{C}^3$ as complex manifolds, though not as complex Lie groups because of the different group structure. Take the subgroup $\Gamma = G \cap GL(3, \mathbb{Z} + i\mathbb{Z})$; it acts properly discontinuously on G by left matrix multiplication. Then $X := G/\Gamma$ is a complex 3-fold. The 1st and 3rd coordinates give a holomorphic map $f : X \to \mathbb{C}/(\mathbb{Z}+i\mathbb{Z}) \times \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ wich fibers isomorphic to $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$. Then locally, it looks like three copies of an elliptic curve (which is not T^6).

Example 2.9. Godeaux surface. Let ρ be a primitive 5th root of unity. For simplicity, say $\rho = \exp(2\pi i/5)$. Then $G = \langle \rho \rangle \cong \mathbb{Z}_5$ acts on $\mathbb{P}^3$ by $\rho \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : \rho z_1 : \rho^2 z_2 : \rho^3 z_3]$. This has 4 fixed points: $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1]$.

Consider the surface Y in $\mathbb{P}^3$ defined by $\sum_{j=0}^3 z_j^5 = 0$. It is G-invariant and since the fixed points are not in Y, the action on Y is fixed-point free. $X = Y/G$ is a compact complex surface and it can be shown that $H^i(X, \mathcal{O}) = 0$ for $i = 1, 2$ but is also not rational.

Example 2.10. Let $G \cong \mathbb{Z}_5$ be the 5th roots of unity and let $\widetilde{G} \subset G^5$ be a subgroup defined by

$$\widetilde{G} = \left\{ (\xi_0, ..., \xi_4) : \xi_j \in G, \prod_{j=0}^{4} \xi_j = 1 \right\}.$$

This subgroup has $625 = 5^4$ elements because the first four entries can be anything so long as the 5th is the inverse of the product of the previous four. It acts on $\mathbb{P}^4$ by $(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) \cdot [z_0 : z_1 : z_2 : z_3 : z_4] = [\xi_0 z_0 : \xi_1 z_1 : \xi_2 z_2 : \xi_3 z_3 : \xi_4 z_4]$. The subgroup $H \subset \widetilde{G}$ which acts trivially on $\mathbb{P}^4$ is just the diagonal isomorphic to $\mathbb{Z}_5$. The hypersurface X_t, $t \in \mathbb{C}$ defined by

$$\sum_{j=0}^{4} z_j^5 - 5t \prod_{j=0}^{4} z_j$$

is invariant under $\widetilde{G}$. So the solutions to this equation should satisfy $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 5t z_0 z_1 z_2 z_3 z_4$. For example, take $[1 : -1 : 0 : 0 : 0]$. Then note that it is stabilized by nontrivial elements in $\widetilde{G}/H$ such as $(1, 1, \rho, \rho, \rho^2)$. Then $Y_t = X_t/(\widetilde{G}/H)$ is not a manifold but has some rather mild singularities.

3 Almost Complex Geometry

Example 3.1. Let X be a closed, smooth 6-manifold. As it turns out, the only obstruction to X being almost complex is whether it is $Spin^c$. That is, we only need the 2nd Stiefel-Whitney class $w_2 \in H^2(X, \mathbb{Z}_2)$ to be the mod 2 reduction of some class $c \in H^2(X, \mathbb{Z})$. If this happens, then in fact, each preimage of w_2 under this mod 2 reduction is realized as the 1st Chern class c_1 of some almost complex structure J on TX.

Claude LeBrun showed in his paper *Topology versus Chern Numbers for Complex 3-Folds* that the Chern numbers c_1^3 and $c_1 c_2$ of a complex 3-fold are not determined by the underlying topology of the smooth 6-manifold.

As it turns out, if we think of $\mathbb{R}^7$ as the imaginary octonions, then any orientable hypersurface of $\mathbb{R}^7$ is almost complex. To define the J, we take advantage of the octonion structure.

4 Kähler Geometry

By Hodge Theory, we can see that for a Kähler manifold, $h^{p,q} = h^{q,p}$. Also if we consider some odd cohomology H^{2k+1}, then the number of pairs (p, q) such that $p + q = 2k + 1$, is even in number. These two facts together show that the odd Betti numbers of a Kähler manifold are even.

Example 4.1. [Kodaira Embedding Theorem]

Roughly speaking, a holomorphic line bundle $L \to X$ is **very ample** if has an abundance of holomorphic sections; enough sections so that we can use them to define an embedding of X into some large $\mathbb{C}P^N$. Intuitively, say we have N holomorphic sections, locally on $U \subset X$, these are N functions $U \to \mathbb{C}$ which together give a map $X \to \mathbb{C}^N$. If all goes well, perhaps $U \to \mathbb{C}^N$ is an embedding (no singularities in the intersection of these sections) and we can hope to patch these together to get an embedding $X \hookrightarrow \mathbb{C}P^N$.

A line bundle is **ample** if some high enough tensor power is very ample: $L^{\otimes k}$ is very ample. So then, we could have sections $s_0, ..., s_N \in H^0(X, L^{\otimes k})$ and an embedding $X \to \mathbb{C}P^N$ which sends $x \mapsto [s_0(x) : ... : s_N(x)]$.

Kodaira's Embedding Theorem says that if the Kähler form ω of a compact Kähler manifold, which lives in $H^2(X;\mathbb{R}) \cap H^{1,1}(X,\mathbb{C})$ is in fact **integral**, that is $\omega \in H^2(X;\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$, then there is a complex-analytic embedding of X into some large complex projective space.

A stronger view of the theorem is as follows: Let X be a compact Kähler manifold and $L \to X$ be a holomorphic line bundle. It is ample if and only if there is a holomorphic embedding $\varphi : X \to \mathbb{C}P^N$ such that $\varphi^*\mathcal{O}_{\mathbb{C}P^N}(1) = L^{\otimes k}$ for some $k > 0$.

This means that the existence of an ample line bundle is equivalent to whether a compact Kähler manifold is projective with an additional condition.

We may also relate this to differential geometry with curvature. A holomorphic line bundle $L \to X$ is called **positive** if $c_1(L) \in H^2(X,\mathbb{R})$ is representable by a closed, positive definite, real $(1,1)$-form.

The Kodaira embedding theorem implies that a positive line bundle is an ample line bundle and conversely that any ample line bundle admits a Hermitean metric that makes it a positive line bundle.

Example 4.2. An amazing thing in the case of surfaces, for example, is suppose we have a compact complex surface X. Then X is Kähler if and only if b_1 is even. This is a quite a powerful theorem and is an overpowered tool for showing, for example, that S^4 is not complex. $b_1(S^4) = b_2(S^2) = 0$ which means it cannot be symplectic and hence, is not complex to begin with.

Example 4.3. A natural question to ask in the setting of compact Kähler manifolds X, is: "What is the relationship between the 1st Chern class c_1 and Kähler form ω of X?"

In the case of $\mathbb{C}P^n$, we know that the even cohomology $H^{2k}(X,\mathbb{C})$ is generated by ω^k and the cohomology ring is $H^*(\mathbb{C}P^n,\mathbb{Z}) = \mathbb{Z}[x]/x^{n+1}$. But also $c_1 \in H^2(X,\mathbb{Z})$. Is ω an integral class? We usually take the Fubini-Study form: $\omega = \frac{i}{2}\partial\overline{\partial}\log|z|^2$; I think the constants are chosen to make this an integral class.

Jiahao says that in general, there is a relationship between the curvature of and the 1st Chern class. If h is the Hermitian metric for which ω is the Kähler form, then $\frac{1}{2\pi}\mathrm{Ric}(h) = c_1(X)$. So there's still some relationship between c_1 and curvature. Chern-Weil theory tells us, for example, that if we consider the curvature of **any** connection and plug this into some invariant polynomial, we ultimately get the some cohomology class which does not depend on the connection.

Note: The cohomology ring of $\mathbb{C}P^\infty$ is $H^*(\mathbb{C}P^\infty,\mathbb{Z}) = \mathbb{Z}[x]$. However, observe that $\lim_{\leftarrow} H^*(\mathbb{C}P^n,\mathbb{Z}) = \mathbb{Z}[\![x]\!]$; this is the ring of formal power series which is not the same as the polynomial ring $\mathbb{Z}[x]$. So here is an example of when cohomology does not commute with inverse limit.

Example 4.4. Claim: Any compact complex submanifold in a Kähler manifold is volume minimizing within its homology class. Note that this does not say the minimizer is unique. Indeed, if we have a second volume minimizing submanifold in the same class as the given complex submanifold, that submanifold must also be complex.

To establish this, we need to begin with a discussion of calibrated geometry. A **calibrated manifold** is a Riemannian manifold (M^n, g) equipped with a k-form φ $(1 \leq k \leq n)$ such that $d\varphi = 0$ and for any $x \in M$ and any oriented k-dim subspace $P \subset T_xM$, $\varphi|_P = \lambda vol_P$, $\lambda \leq 1$. Of course, the volume element is determined by g.

Now, let $G_x(\varphi) = \{P : \varphi|_P = vol_P\}$; this is the space of k-planes of T_xM on which φ restricts precisely to the volume element. Let $G(\varphi) = \bigcup_{x \in M} G_x(\varphi)$.

A submanifold $N \subset M$ is **calibrated** with respect to φ if $TN \subset G(\varphi)$. We can easily show that a calibrated submanifold is volume minimizing in its real homology class. Let N be calibrated and N' be another submanifold in its homology class. This tells us that N' differs from N in homology by a boundary. Then,

$$\mathrm{vol}(N) = \int_N vol_N = \int_N \varphi = \int_{N'} \varphi + \int_{\partial L} \varphi = \int_{N'} \varphi = \lambda \int_{N'} vol_{N'} \leq \mathrm{vol}(N').$$

The fourth equality holds by Stokes theorem and the fifth because φ is a calibration.

Let (X, ω) be a Kähler manifold. Wirtinger's inequality says that $\omega^k(Z) \leq k!$ for any $2k$-vector Z; so $\omega^k/k! \leq 1$ which means $\omega^k/k!|_P = \lambda vol_P$ where $\lambda \leq 1$. So $\omega^k/k!$ is a calibration.

On the other hand, if $Y \subset X$ is a compact, complex k-submanifold, Wirtinger's formula shows that

$$\mathrm{vol}(Y) = \int_Y \frac{\omega^k}{k!}.$$

Hence, Y is a **calibrated submanifold** and therefore, a volume minimizer in its real homology class. I'm not sure how to prove the other statement.

5 Deformation Theory

The main question of deformation theory: "Is X diffeomorphic to Y if and only if X is biholomorphic to Y?"

From math.overflow or somewhere... "Of course, manifolds which are biholomorphic are diffeomorphic. But the converse is false. As Riemann was the first to remark, and others after him did until the foundational works of Kodaira and Spencer in the 60's, complex structures on manifolds come many at a time. Unlike discrete invariants such as dimension or "number of holes," complex structures on a fixed manifold form a continuous space. Thus, if a fixed manifold X admits one complex structure, then in general it admits an entire family of complex structures. Thus you get infinitely many complex manifolds X_s, depending on some parameter s, which will all be diffeomorphic as smooth manifolds. The converse implication of your question thus fails dramatically."

Example 5.1. Let us consider elliptic curves. Let $s \in \mathbb{H} = \{s \in \mathbb{C} : \mathrm{Im}\ s > 0\}$. Each such point determines an elliptic curve $X_s = \mathbb{C}/\Lambda_s$, where Λ_s is the lattice $\mathbb{Z} \oplus s\mathbb{Z}$. When are two such curves biholomorphic?

By lifting a potential biholomorphism $f : X_s \to X_{s'}$ to a map $\tilde{f} : \mathbb{C} \to \mathbb{C}$, one can prove that f must be induced by a linear map $\tilde{f}$ that sends the lattice Λ_s to $\Lambda_{s'}$. Now these maps are very rare. For example, for ϵ small enough, the lattices $\mathbb{Z} \oplus i\mathbb{Z}$ and $\mathbb{Z} \oplus i(1+\epsilon)\mathbb{Z}$ will never be isomorphic (for a heuristic reason: holomorphic maps cannot stretch the y-axis while keeping the x-axis fixed). Thus we get infinitely many non-biholomorphic elliptic curves X_s.

On the other hand, it is easy to see that any elliptic curve is diffeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$. Indeed, there is a unique $\mathbb{R}$-linear map of $\mathbb{R}^2$ to itself sending the lattice $\mathbb{Z}^2$ to any lattice of our choice. This map will induce a diffeomorphism of the corresponding elliptic curves.

6 Classification of Compact Complex Surfaces

Here is a quick summary of the Enriques-Kodaira classification of compact complex surfaces. There are 10 types (which include non-algebraic types) which break up into 4 larger groups based on Kodaira dimension.

1. Kodaira dimension $-\infty$:
   - Rational: surfaces birational to $\mathbb{C}P^2$.
   - Ruled (genus > 0): always algebraic.
   - Type VII: never algebraic or Kähler. e.g. when $b_2 = 0$: Hopf and Inoue surfaces.
2. Kodaira dimension 0:
   - K3: always Kähler, not always algebraic.
   - Enriques: always algebraic.
   - Kodaira: never algebraic.
   - Complex 2-Tori: always Kähler, not always algebraic.
   - Hyperelliptic: always algebraic; these are quotient of a product of two elliptic curves by a finite group.
3. Kodaira dimension 1: all such surfaces are **elliptic**; i.e. have an elliptic fibration. However, some elliptic surfaces have Kodaira dimension 0. e.g. some K3's.
4. Kodaira dimension 2: They are called general type surfaces and are always algebraic.