Classical Field Theory – Charles Torre

A mechanical system is a dynamical system w/ finitely many degrees of freedom.

A field is also a dynamical system but w/ infinitely many degrees of freedom.

Mathematically, a field is a section of a fiber bundle.

Let $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be a scalar field

$x^\alpha = (t, x, y, z)$

The Klein-Gordon Equation is

$$\Box \varphi - m^2 \varphi = 0$$

where $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$

a wave operator called the d'Alembertian

When $m=0$, we get the wave eqn:

$$\underset{\substack{\uparrow \\ \text{Laplacian}}}{\Delta} \varphi = \partial_t^2 \varphi$$

The KG eqn came about as an attempt to give a relativistic Schrödinger eqn but this did not work; afterall, Schrödinger is mechanical, i.e. finitely many deg of freedom

But KG is sort of a classical limit of a quantum field in relativistic settings

How to solve KG? Suppose φ is well behaved, such as

$$\varphi_t \in L^2(\mathbb{R}^3) \quad \forall t.$$

Taking a Fourier expansion:
$$\varphi(t,r) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} \hat{\varphi}_k(t)\, e^{ik\cdot r}\, d^3k$$

We have $\hat{\varphi}_{-k} = \hat{\varphi}_k^*$ since φ is real valued.

If φ satisfies $\Box\varphi = m^2\varphi$, then

$$\int -\partial_t^2\left(\hat{\varphi}_k(t)\right)e^{ik\cdot r}\,d^3r$$

$$+\int \partial_x^2\left(\hat{\varphi}_k e^{ik\cdot r}\right) + \partial_y^2\left(\hat{\varphi}_k e^{ik\cdot r}\right) + \underbrace{\partial_z^2\left(\hat{\varphi}_k e^{ik\cdot r}\right)}_{\shortparallel}\,d^3r$$

$$= m^2\int \hat{\varphi}_k \cdot e^{ik\cdot r}\,d^3r$$

$$\hat{\varphi}_k \cdot \partial_z^2\left(e^{ik\cdot r}\right) = -k_z^2\,\hat{\varphi}_k\,e^{ikr}$$

$\Rightarrow$ In the integrands:

$$\left(-\ddot{\hat{\varphi}}_k - \vec{k}\cdot\vec{k}\,\hat{\varphi}_k\right)e^{ik\cdot r} = m^2\hat{\varphi}\,e^{ik\cdot r}$$

dot product

$$\Rightarrow \ddot{\hat{\varphi}}_k + (k^2 + m^2)\varphi_k = 0.$$

norm square

This is easy to solve; it's just a 2^{nd} order ODE of a simple form.

$$\hat{\varphi}_k(t) = a_k e^{i\omega_k t} + b_k e^{-i\omega_k t}$$

$$\omega_k = \sqrt{k^2 + m^2}$$

Since $\hat{\varphi}_{-k} = \hat{\varphi}_k^*$, then $b_{-k} = a_k^*$.

$$\therefore \varphi(k) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} \left(a_k e^{i(k\cdot r - \omega_k t)} + a_k^* e^{-i(k\cdot r - \omega_k t)} \right) d^3 r$$

KG field

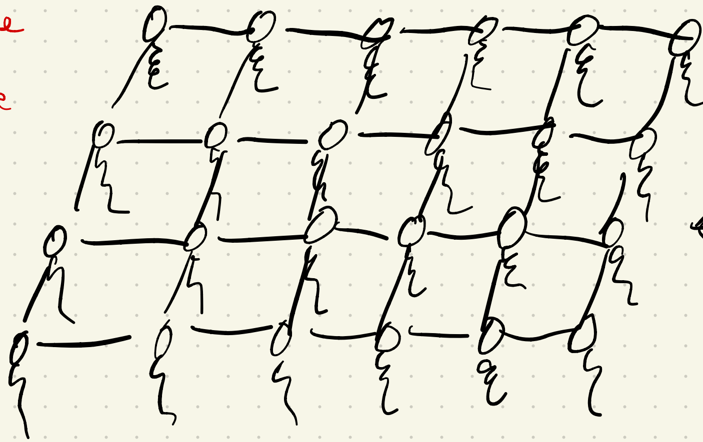
Note: The solution φ to KG is essentially an infinite collection of uncoupled harmonic oscillators for each $k \in \mathbb{R}^3$

So there are infinite deg of freedom

Also, the KG field is often called the free or non-interacting field b/c of the uncoupled nature of the oscillators

So picture an array of springs as a visual of a field φ satisfying KG

Discrete picture



nudge it here to "excite" the array. Then a wave will propagate outwards.

This excitation which leads to a wave is what we call a particle. Particles are waves in QM.

Let $L = \frac{1}{2}\int_{\mathbb{R}^3} \underbrace{\left(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2\right)}_{= \mathcal{L} \text{ (Lagrangian density)}} d^3x$ (Lagrangian on space)

$S[\varphi] = \int_{t_1}^{t_2} L\, dt$, (action), let $R = [t_1, t_2] \times \mathbb{R}^3$

include time now

Variation of S.

Let $\lambda \in \mathbb{R}$ be a parameter & φ_λ a 1-parameter family of fields. w/ $\varphi_0 = \varphi$.

$$\delta S = \left.\frac{d S[\varphi_\lambda]}{d\lambda}\right|_{\lambda=0}. \quad \text{Also } \delta\varphi \doteq \left.\frac{d\varphi_\lambda}{d\lambda}\right|_{\lambda=0}.$$

Then $$\delta S = \int_R \left(\dot{\varphi}\,\delta\dot{\varphi} - \nabla\varphi\cdot\nabla\delta\varphi - m^2\varphi\,\delta\varphi\right) d^4x.$$

Note: $\nabla\cdot(\nabla\varphi\,\delta\varphi) = \nabla^2\varphi\,\delta\varphi + \nabla\varphi\cdot\nabla\delta\varphi.$

Using integration by parts, note:

$$\int \dot{\varphi}\,\delta\dot{\varphi} = \dot{\varphi}\,\delta\varphi - \int \ddot{\varphi}\,\delta\varphi.$$

$$\therefore \delta S = \int_R (-\ddot{\varphi} + \nabla^2\varphi - m^2\varphi)\,\delta\varphi\, d^4x + \left[\int_{\mathbb{R}^3} \dot{\varphi}\,\delta\varphi\right]_{t_1}^{t_2}$$

$$\underbrace{- \int_R \nabla\cdot(\nabla\varphi\,\delta\varphi)\, d^4x.}_{= -\int_{t_1}^{t_2} dt \int_{r\to\infty} n\cdot\nabla\varphi\,\delta\varphi\, d^2A}$$

(Divergence Theorem)

If φ has cpt support or $\to 0$ faster than $\frac{1}{r^2}$, then the last term (boundary term) vanishes.

Other boundary conditions, such as the endpoints of φ_i are fixed will make $\delta\varphi|_{t_1} = \delta\varphi|_{t_2} = 0$.

This will force the middle term to vanish.

$\therefore$ w/ all these conditions,

$$\delta S = 0 \text{ when } \int_R (-\ddot{\varphi} + \nabla^2\varphi - m^2\varphi)\,\delta\varphi\, d^4x = 0$$

$\Rightarrow \Box\varphi - m^2\varphi = 0$ everywhere in R.

So S is the correct action for KG eqn

Another view is via Euler-Lagrange eqn.

$$\mathcal{L} = \frac{1}{2}\left(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2\right)$$

Treating $x, \varphi, \varphi_\alpha$ as formal variables (so $\mathcal{L}$ is in the 1st Jet space)

define $$E(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \varphi} - D_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_\alpha}$$

total derivative

Then $\dfrac{\delta S}{\delta \varphi} = \mathcal{E}(\mathcal{L})\Big|_{\varphi=\varphi(x)} = \Box\varphi - m^2\varphi.$

$\in J^2$ (2nd jet space)

In fact, if $\hat{\mathcal{L}} = \frac{1}{2}\varphi(\Box - m^2\varphi)$, one can show

$$\mathcal{E}(\mathcal{L}) = \mathcal{E}(\hat{\mathcal{L}}) \quad \& \quad \mathcal{L} = \hat{\mathcal{L}} + \text{divergence term}$$

sum of all the 1st partial derivatives of something.

Some Generalizations of KG.

Let $\mathcal{L} = \frac{1}{2}(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2) - j\varphi$

where $j : \mathbb{R}^4 \to \mathbb{R}$ is the "source" (in electromagnetism, j could be electric charge or current)

Then $\mathcal{E}(L) = (\Box - m^2)\varphi - j.$

In QFT, the presence of a source leads to particle creation/annihilation via transfer of energy-momentum b/w the field & its source.

The KG eqns are linear; so the solutions are non-interacting.

If we modify KG to be non-linear, we introduce self-interaction.

potential

$$\text{E.g. } \mathcal{L} = \frac{1}{2}\left(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2\right) - V(\varphi)$$

$$E(L) = (\Box - m^2)\varphi - V'(\varphi) = 0.$$

So long as V isn't quadratic, this becomes non-linear.

One potential of interest: $V(\varphi) = -\frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4$.

If V is quad, then $V'(\varphi) = a\varphi - b$

; so $\left(\Box - (m^2+a)\varphi\right) = b$

Seems like, you just increase the mass... what does this mean, physically?

Coordinate free description: Let (M,g) be a Lorentzian mfd.

Then $\mathcal{L} = -\frac{1}{2}\left(g^{-1}(d\varphi, d\varphi) + m^2\varphi^2\right)\varepsilon(g)$

volume form of g.

is the Lagrangian density.

This is called minimally coupled

Let ξ be a parameter & $R(g)$ = scalar curvature. Then

$$\mathcal{L} = -\frac{1}{2}\left[g^{-1}(d\varphi, d\varphi) + (m^2 + \xi R(g))\varphi^2\right]\varepsilon(g)$$ gives

Curvature coupled KG theory.

These theories are not "diffeomorphism invariant" or "generally covariant"; i.e. $f: M \xrightarrow{\text{diffeo}} M$ & $\hat{g} = f^* g$, then

$$\hat{\mathcal{L}} = -\frac{1}{2}\left(\hat{g}^{-1}(d\varphi, d\varphi) + m^2\varphi^2\right)\varepsilon(\hat{g})$$ is a new

Lagrangian density in general unless f is a symmetry.

If we allow g to vary, we get 11 coupled non-linear field eqns instead of 1 linear field eqn.

Conservation Laws are fundamental & give info about complicated dynamics. Also, conservation laws are related to symmetries by Noether's theorem.

def: Let $j^\alpha = j^\alpha(x, \varphi, \partial\varphi, \ldots, \partial^k\varphi) \in \mathcal{J}^k$ be a vector field constructed as a local f^n. j^α is a conserved current or defines a conservation law if the divergence of $j^\alpha = 0$ when φ satisfies the field eqn.

e.g. $D_\alpha j^\alpha = 0$ when $(\Box - m^2)\varphi = 0$.

Explicitly, if $\varphi(x)$ is a solution, take

$$j^\alpha(x) \doteq j^\alpha\left(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \ldots, \frac{\partial^k\varphi(x)}{\partial x^k}\right) \quad \text{s.t.}$$

$$\frac{\partial}{\partial x^\alpha} j^\alpha = 0.$$

If $j^\alpha = (j^0, j^1, j^2, j^3)$, call $\rho \doteq j^0$ the density

$\vec{j} = (j^1, j^2, j^3)$ the current density

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0.$$

The point of writing this is:

Let $Q_V(t) = \int_V \rho(t, \vec{x})\, d^3x$ be the total charge in region V.

Then $$\frac{d}{dt} Q_V(t) = -\int_V \nabla \cdot \vec{j} = \underbrace{-\int_{\partial V} \vec{j}\, d^2S}_{\text{net flux}}$$

We say Q_V is conserved since we can see how it changes over time by purely in terms of the net flux, a fixed value. So there's no creation nor destruction of charge; it just moves around.

If we place boundary conditions, such as $V = \mathbb{R}^3$; the field vanishes rapidly enough at ∞, then $\frac{d}{dt} Q_V(t) = 0$; so the total charge is constant

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Conservation of Energy

Let $j^0 = \frac{1}{2}(\dot{\varphi}^2 + |\nabla\varphi|^2 + m^2\varphi^2)$

$j^i = -\dot{\varphi}(\nabla\varphi)_i$

$$\left.\begin{aligned} \partial_0 j^0 &= \dot{\varphi}\ddot{\varphi} + \nabla\varphi\cdot\nabla\dot{\varphi} + m^2\varphi\cdot\dot{\varphi} \\ \underbrace{\partial_i j^i}_{} &= -\nabla\dot{\varphi}\cdot\nabla\varphi - \dot{\varphi}\nabla^2\varphi \end{aligned}\right\} \Rightarrow \partial_\alpha j^\alpha = -\dot{\varphi}(\Box\varphi - m^2\varphi)$$

Sum the Einstein notation

So if φ is a solution to KG, $\partial_\alpha j^\alpha = 0$.

Let $E_V = \frac{1}{2}\int_V \left(\dot{\varphi}^2 + |\nabla\varphi|^2 + m^2\varphi^2\right) d^3x$

total energy

$$= T + U$$

$$T = \frac{1}{2}\int_V \dot{\varphi}^2\, d^3x \qquad U = \frac{1}{2}\int_V \left(|\nabla\varphi|^2 + m^2\varphi^2\right) d^3x$$

kinetic energy $\qquad$ potential energy

Then $\qquad \rho = j^0 \qquad \vec{j} = (j^1, j^2, j^3)$

$$\frac{d}{dt} E_V = \int \partial_t \rho = -\int \nabla\cdot\vec{j} = -\int_V \overbrace{\left(\nabla\varphi\cdot\nabla\dot{\varphi} + \dot{\varphi}\nabla^2\varphi\right)}^{\nabla \cdot (\dot{\varphi}\nabla\varphi)} d^3x$$

$$= -\int_{\partial V} \dot{\varphi}\,\nabla\varphi\, dS.$$

If $\dot{\varphi} = 0$ or the normal component of $\nabla\varphi$ to ∂V vanishes,

then $\frac{d}{dt} E_V = 0$. {Energy is conserved.

Energy - Momentum Tensor (p. 49)
(or Stress - Energy tensor) $g = \mathrm{diag}(-1,1,1,1)$

Given $\varphi : (\mathbb{R}^4, g) \to \mathbb{R}$, the energy-momentum tensor is defined as

$$T = d\varphi \otimes d\varphi - \frac{1}{2}\left(g^{-1}(d\varphi, d\varphi) - m^2\varphi^2\right) g$$

(it's symmetric)

Its components: $T_{\alpha\beta} = T_{\beta\alpha}$

$$T_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta}\varphi_{,\gamma}\varphi_{,\delta} - \frac{1}{2} m^2\varphi^2 g_{\alpha\beta}$$

Then $$J^{\alpha}_{\text{energy}} = -T_t^{\ \alpha} \equiv -g^{\alpha\beta} T_{t\beta}$$

; energy density $= T^{tt}$

$$J^{\alpha}_{\text{momentum}} = -T_i^{\ \alpha} \equiv -g^{\alpha\beta} T_{i\beta}, \quad i = 1,2,3$$

Momentum density in direction i is $-T^{ti}$

Conservation of energy & momentum is encoded in an identity:

$$g^{\beta\gamma}\partial_\gamma T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi$$

where $\Box\varphi = g^{\alpha\beta}\varphi_{,\alpha\beta}$.

So if $(\Box - m^2)\varphi = 0$, then

$$g^{\beta\gamma}\partial_\gamma T_{\alpha\beta} = 0$$

i.e. the divergence of the energy-momentum tensor vanishes

Note: Change of reference frame mixes up energy & momentum. So one often says, "Conservation of energy-momentum", not just one of these.

A symmetry $f: \mathbb{R}^4 \to \mathbb{R}^4$ is a diffeo s.t. if $\hat{\varphi} = \varphi \circ f$, then the Lagrangian is preserved.

So if

$$\mathcal{L}(x, \hat{\varphi}, \partial\hat{\varphi}) = \mathcal{L}(x, \varphi, \partial\varphi)$$

then the Lagrangian is preserved. There is a more general notion of symmetry called divergence symmetry.

Since, if $\hat{\mathcal{L}} = \mathcal{L} + (\text{div term})$, then

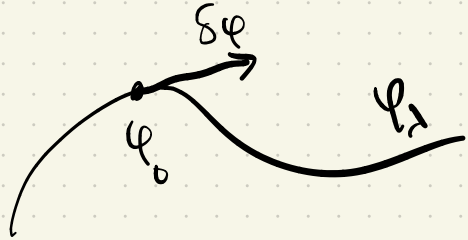
$\mathcal{E}(\hat{\mathcal{L}}) = \mathcal{E}(\mathcal{L})$, diffeos which change the Lagrangian by a divergence term are also considered to be symmetries.

We typically study 1-param families of symmetries, say $f_\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ & let $\varphi_\lambda \doteq \varphi \circ f_\lambda$. Then

we want $$\frac{d\mathcal{L}(x, \varphi_\lambda, \partial_x \varphi_\lambda)}{d\lambda} = 0.$$

Infinitesimal Symmetry (a vector field) p. 55

Space of fields



– path of fields defined by continuous symmetries

$$\delta\varphi \doteq \left.\frac{d\varphi_\lambda}{d\lambda}\right|_{\lambda=0}.$$

This $\delta\varphi$ is an infinitesimal symmetry.

View it as a vect field on the space of fields

Since Lagrangian densities which differ by a divergence term gives the same Euler-Lagrange eqns, then we expand our definition of symmetry to also include $f:\mathbb{R}^4\to\mathbb{R}^4$ ($\hat{\varphi} = \varphi\circ f$) s.t.

$$\mathcal{L}(x,\hat{\varphi},\partial_x\hat{\varphi}) = \mathcal{L}(x,\varphi,\partial_x\varphi) + \partial_\alpha V^\alpha.$$

e.g. Consider time translation:

$$\varphi(t,x) \longmapsto \varphi(t+\lambda, x)$$

means partial derivatives wrt α & β

Then $$\mathcal{L} = -\frac{1}{2}\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + m^2\varphi^2\right)$$

$g^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$

Then $$\delta\mathcal{L} = -\left(g^{\alpha\beta}\varphi_{,\alpha}\dot{\varphi}_{,\beta} + m^2\varphi\dot{\varphi}\right)$$

$$= \partial_t\mathcal{L}$$

$$= \partial_\alpha(\delta^\alpha_t\mathcal{L}).$$ ← Divergence term

This means there is no preferred instant of time in KG theory

Noether's Thm

Let $\mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi)$. If $\delta\varphi$ is an infinitesimal variational symmetry, then $\delta\mathcal{L} = 0$.

But also, at any pt in the space of fields

$$\delta\mathcal{L} = \mathcal{E}(\mathcal{L})\,\delta\varphi + D_\alpha V^\alpha$$

where $\mathcal{E}(\mathcal{L}) = \dfrac{\partial\mathcal{L}}{\partial\varphi} - D_\alpha\left(\dfrac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\right)$

& $V^\alpha = \dfrac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\,\delta\varphi$

holds for any field variation

So if $\delta \mathcal{L} = 0$, then $D_\alpha V^\alpha = -\mathcal{E}(\mathcal{L})\delta\varphi$

This is exactly the type of identity needed for defining a conserved current V^α: if φ satisfies the KG eqn, then $\mathcal{E}(\mathcal{L}) = 0$ & so

$$D_\alpha V^\alpha = 0.$$

Or if $\delta \mathcal{L} = D_\alpha W^\alpha$, then

$$D_\alpha (V^\alpha - W^\alpha) = -\mathcal{E}(\mathcal{L})\delta\varphi$$

& ~~the~~ conserved current is $V^\alpha - W^\alpha$.

Noether's First Thm:

If $\delta\varphi(x,\varphi,\partial\varphi,\ldots)$ is a divergence symmetry of $\mathcal{L}(x,\varphi,\partial\varphi)$; i.e.

$$\delta\mathcal{L} = D_\alpha W^\alpha,$$

then there exists a conserved current given by

$$j^\alpha = \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\delta\varphi - W^\alpha.$$

Application: the time translation symmetry gives conserved current defining conservation of energy

Recall the time translation is a divergence symmetry:

$$\delta\varphi = \dot{\varphi} \Rightarrow \delta\mathcal{L} = D_\alpha(\delta^\alpha_t \mathcal{L})$$

Recognizing that $\delta\varphi = \dot\varphi$, $W^\alpha = \delta^\alpha_t \mathcal{L}$, ;

$$\frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} = -g^{\alpha\beta}\varphi_{,\beta},$$

we have: $j^\alpha = -g^{\alpha\beta}\varphi_{,\beta}\varphi_{,t} - \delta^\alpha_t \mathcal{L} = -T^\alpha_t$

This j^α is a conserved current. We did the calculation $D_\alpha j^\alpha = 0$ earlier when discussing conservation of energy.

We sometimes say the conserved current for energy is the Noether current associated to time translation symmetry

This was predictable since $\mathcal{L}$, in the jet space, has no dependence on t, only on $\varphi, \partial\varphi$.

Space translations & Conservation of Momentum

Let $\hat{n}$ be a const unit vect field on $\mathbb{R}^3$. The space translation is given by $\varphi(t,\vec{x}) \longmapsto \varphi_\lambda(t,\vec{x}) \doteq \varphi(t,\vec{x}+\lambda\hat{n})$

$\vec{x} = (x,y,z)$

Then
$$\delta\varphi \doteq \left.\frac{d\varphi_\lambda}{d\lambda}\right|_{\lambda=0} = \partial_x \varphi_\lambda(t,\vec{x})\cdot \hat{n}_x + \partial_y \varphi_\lambda(t,\vec{x})\cdot \hat{n}_y + \partial_z \varphi_\lambda(t,\vec{x})\cdot \hat{n}_z$$

$$= \hat{n}\cdot\nabla\varphi_\lambda(t,\vec{x})\Big|_{\lambda=0}$$

$$= \hat{n}\cdot\nabla\varphi.$$

$$= n_i \cdot \varphi_{,i}$$

To check it is a symmetry, compute

$$\delta\mathcal{L} = \delta\left(\tfrac{1}{2}\left(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2\right)\right)$$

$$= \dot{\varphi}\,\delta\dot{\varphi} - \nabla\varphi\cdot\nabla\delta\varphi - m^2\varphi\cdot\delta\varphi$$

$$= \dot{\varphi}(\hat{n}\cdot\nabla\dot{\varphi}) - \nabla\varphi\cdot\nabla(\hat{n}\cdot\nabla\varphi) - m^2\varphi(\hat{n}\cdot\nabla\varphi)$$

$\hat{n}$ is const

$$= \hat{n}\left(\dot{\varphi}\cdot\nabla\dot{\varphi} - \nabla\varphi\cdot\nabla^2\varphi - m^2\varphi\cdot\nabla\varphi\right)$$

On the other hand:

$$\nabla \mathcal{L} = \dot{\varphi}\, \nabla \dot{\varphi} - \nabla\varphi \cdot \nabla^2 \varphi - m^2 \varphi \nabla \varphi$$

So $\delta \mathcal{L} = \hat{n} \cdot \nabla \mathcal{L} = \nabla (\hat{n} \cdot \mathcal{L})$

$$= \underbrace{D_\alpha}_{\text{total derivative}} (W^\alpha) \quad \text{where} \quad W^\alpha = (0, n^i \mathcal{L})$$

So space translation $=$ divergence symmetry.

Then, $j^\alpha = (\rho, j^i) = \dfrac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} \delta\varphi - W^\alpha$ by Noether's Thm

So $\rho = \dot{\varphi}\, \delta\varphi - 0 = \dot{\varphi}\, \hat{n} \cdot \nabla \varphi$.

Since $|\nabla\varphi|^2 = \varphi_{,1}^2 + \varphi_{,2}^2 + \varphi_{,3}^2$, then

$$j^i = -\varphi_{,i}\, \hat{n} \cdot \nabla\varphi - \frac{n^i}{2} \mathcal{L}$$

$\rho = \dot{\varphi}\, \hat{n}\cdot\nabla\varphi\ ,\quad j^i = -\varphi_i\, \hat{n}\cdot\nabla\varphi + \frac{1}{2} n^i \left(|\nabla\varphi|^2 - \dot{\varphi}^2 + m^2\varphi^2\right)$

Let's verify $\frac{d\rho}{dt} + \nabla\cdot\vec{j} = 0$.

$$\frac{d\rho}{dt} = \ddot{\varphi}\, \hat{n}\cdot\nabla\varphi + \dot{\varphi}\, \hat{n}\cdot\nabla\dot{\varphi}$$

cancel

Once summed over i, these cancel.

$$\partial_i j^i = -\varphi_{ii}\, \hat{n}\cdot\nabla\varphi - \varphi_i\, \hat{n}\cdot\nabla\varphi_i + n^i\left(\nabla\varphi_i\cdot\nabla\varphi - \dot{\varphi}\,\dot{\varphi}_i + m^2\varphi\,\varphi_i\right)$$

$$\Rightarrow \nabla\cdot\vec{j} = -\nabla^2\varphi\cdot\hat{n}\cdot\nabla\varphi - \dot{\varphi}\,\hat{n}\cdot\nabla\dot{\varphi} + m^2\varphi\cdot\hat{n}\cdot\nabla\varphi$$

$$\frac{d\rho}{dt} + \nabla\cdot\vec{j} = \ddot{\varphi}\,\hat{n}\cdot\nabla\varphi - \nabla^2\varphi\cdot\hat{n}\cdot\nabla\varphi + m^2\varphi\,\hat{n}\cdot\nabla\varphi$$

$$= -\underbrace{(\Box - m^2)\varphi}_{=\,0 \text{ since } \varphi \text{ is a KG field.}}\cdot\hat{n}\cdot\nabla\varphi = 0.$$

Since $\hat{n}$ is arbitrary, we actually have 3 indep conservation laws for 3 linearly indep choices of $\hat{n}$.

Angular momentum.

Q: What symmetries conserve angular momentum?

A: Lorentz symmetries.

def. A Lorentz symmetry of $\mathbb{R}^4$ is a linear transf

$$x^\alpha \longmapsto S^\alpha_\beta x^\beta$$

s.t. the quadratic form $g_{\alpha\beta} x^\alpha x^\beta = -t^2 + x^2 + y^2 + z^2$ is invariant.

So $S^\alpha_\gamma S^\beta_\delta g_{\alpha\beta} = g_{\gamma\delta}$ $(*)$

Let $S(\lambda)$ be a 1-param family of Lorentz symm.

s.t. $S^\alpha_\beta(0) = \delta^\alpha_\beta$, $\omega^\alpha_\beta \doteq \left(\frac{\partial S^\alpha_\beta}{\partial \lambda}\right)\Big|_{\lambda=0}$

Differentiating $(*)$ wrt λ we get: $\omega^\alpha_\gamma g_{\alpha\delta} + \omega^\beta_\delta g_{\gamma\beta} = 0$

Define $\omega_{\alpha\beta} = g_{\beta\gamma}\, \omega^{\gamma}_{\alpha}$. Then Lorentz transf are generated by ω iff $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$.

So the Lie algebra consists of 4×4 matrices X s.t. $gXg = -X^T$ $\big(g = \mathrm{diag}(-1,1,1,1)\big)$.

Then $\delta\varphi = \left(\omega^{\alpha}_{\beta}\, x^{\beta}\right)\varphi_{,\alpha}$

$$\delta\mathcal{L} = D_\alpha\left(\omega^{\alpha}_{\beta}\, x^{\beta}\mathcal{L}\right)$$

$$\text{\&}\quad j^\alpha = \omega_{\gamma\delta}\, \underbrace{M^{\alpha(\gamma)(\delta)}}$$

conserved currents associated to relativistic angular momentum

Claim:

All continuous isometries of flat spacetime are contained in the Poincaré group = diffeomorphisms generated by Lorentz transforms & spacetime translations.

Internal Symmetries

These are symmetries on the space of fields, not on spacetime. Eg. $\varphi \longmapsto -\varphi$.

Here unless $m=0$, there are no interesting continuous internal symmetries.

eg $m=0$, $\varphi_\lambda = \varphi + \lambda$

Then $\delta\varphi = 1$; $\delta\mathcal{L} = \dot\varphi\,\delta\dot\varphi - \nabla\varphi \cdot \overset{0}{\overset{\shortparallel}{\nabla\delta\varphi}}$

$$\delta\dot\varphi = \frac{d\dot\varphi_\lambda}{d\lambda}\bigg|_{\lambda=0} = \underset{\underset{D_\alpha W^\alpha}{\shortparallel}}{0}$$

So $j^\alpha = \dfrac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\delta\varphi - \underbrace{W^\alpha}_{\text{const?}}$

Seems $\rho = \dot\varphi$, $j^i = -\varphi_i$ $\Rightarrow$ $\dfrac{d\rho}{dt} + \nabla\cdot\vec{j} = -\Box\varphi = 0$

since $m=0$;

So $\Box\varphi = 0$.

Charged KG Field & its internal symmetries.

A charged KG field is a $\mathbb{C}$-valued f^n

$$\varphi : M \longrightarrow \mathbb{C} \qquad g_{\alpha\beta} = g^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$$

The Lagrangian is now $\mathcal{L} = -g^{\alpha\beta}\left(\varphi_{,\alpha}\varphi^*_{,\beta} + m^2|\varphi|^2\right)$

We can write this Lagrangian as the sum of two $\mathbb{R}$-valued Lagrangians w/ masses $m_1 = m_2 = m$ & fields φ_1, φ_2.

Then the charged field can be written as

$$\varphi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right).$$

So really, we can work w/ real & imaginary parts if we want: φ & φ^*

The Euler-Lagrange eqns are now:

$$\mathcal{E}_\varphi(\mathcal{L}) = (\Box - m^2)\varphi^* = 0$$

$$\mathcal{E}_{\varphi^*}(\mathcal{L}) = (\Box - m^2)\varphi = 0$$

Working on $\mathbb{C}$ gives new deg of freedom which allows us to introduce conserved electric charge. In QFT, we get anti-particles.

The charged Lagrangian now admits the internal continuous symmetry $\varphi_\alpha = e^{i\alpha}\varphi$, $\varphi^*_\alpha = e^{-i\alpha}\varphi^*$.

There are sometimes called phase transformations or rigid $U(1)$ transformations.

See p. 66-7 for the conserved current - The outcome is

$$j^\alpha = -i g^{\alpha\beta}\left(\varphi^* \varphi_{,\beta} - \varphi \varphi^*_{,\beta}\right)$$

Let $V \subset \mathbb{R}^3$ at a fixed time. Then the total $U(1)$ charge

is $Q = i\int_V (\varphi^* \dot{\varphi} - \varphi \dot{\varphi}^*)\, d^3x$

Use this for modelling electric charge in electrodynamics

Can also use it to model charge which interacts w/ neutral

currents in electroweak theory.

Of course, we can do all this for general

groups w/ representations on vect space V.

Consider $\varphi: M \to V$ w/ internal symmetries given

by $r: G \to \mathrm{Aut}(V)$.

Or even, let V be a vector bundle & $\varphi: M \to V$ a

section

Let $G = SU(2)$, $V = \mathbb{C}^2$.

an element $U \in SU(2)$ can be written as

$$U(\theta, n) = \cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) n^i \sigma_i$$

$n = (n^1, n^2, n^3)$, $\sum_{j=1}^{3} (n^j)^2 = 1$ &

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If $U(\lambda)$ is any 1-param family in $SU(2)$, w/ $U(0) = I$

& $\varphi_\lambda = U(\lambda)\varphi$, then

$\delta\varphi = i\tau\varphi$, where τ is a Hermitian, traceless 2×2 matrix

defined by $\tau = \frac{1}{i}\left(\frac{dU}{d\lambda}\right)_{\lambda=0}$

So $i\tau \in \mathfrak{su}(2)$.

Of course, τ is a $\mathbb{R}$ linear combo of the σ_i

Hermitian inner prod on $\mathbb{C}^2$: $(\varphi_1, \varphi_2) = \varphi_1^* \varphi_2$

Of course, it is $U(n)$ invariant. So we build a Lag.

$$\mathcal{L} = -\left[g^{\alpha\beta}(\varphi_{,\alpha}, \varphi_{,\beta}) + m^2(\varphi, \varphi)\right]$$

Claim: for symmetry $\delta\varphi = i\tau\delta$, the assoc conserved current is $j^\alpha = i g^{\alpha\beta}(\varphi^*_{,\beta}\tau\varphi - \varphi^*\tau\varphi_{,\beta})$.

There are 3 indep conserved currents here b/c of τ.

The 3 conserved charges assoc w/ $SU(2)$ symm are called isospin

Claim: The converse to what we wrote as Noether's 1st Thm is true:

For each conservation law for a system of Euler-Lagrange eqns, there is a corresponding symmetry of the Lagrangian.

For many types of field theories, we also have 1-1 correspondence b/w conservation laws & symmetries of the Lagrangian.

(including KG field theory)

Let $M = (\mathbb{R}^{2n}, \omega)$ (phase space) w/ $H: \mathbb{R}^{2n} \to \mathbb{R}$

a Hamiltonian. Let $\gamma: [0,1] \to \mathbb{R}^{2n}$ be a classical traj.

Then for any function F, $\{F, H\} = \frac{d}{dt} F(\gamma(t))$.

This is b/c we define $\{F, H\} \doteq -(\nabla F)^T J_0 \nabla H$

$$= (\nabla F)^T \dot{\gamma}$$

$$= \frac{d}{dt}(F \circ \gamma)$$

More generally, since $\{\,,\,\}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$,

take another function G; we can think of

G as the infinitesimal generator of a transformation:

$$\delta F = \{F, G\}$$

Then, if $\delta H = \{H, G\} = 0$, we call G a symmetry

of the Hamiltonian system.

But $\dot{G} = \{G, H\} = -\{H, G\} = 0$

So G is also a conserved quantity along class. traj.

i.e. A f^n on phase space generates a symmetry iff it is conserved.