Brief Notes on Coherent Sheaves

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1 Introduction

The general philosophy behind sheaves is as follows. In order to study a topological space, we can study the space itself or we can approach it by studying what sort of functions the space admits. For example, if we have a compact complex manifold, the only global holomorphic functions it admits are constant. Sheaves contain such global data but also local data.

As such, roughly speaking, one can think of sheaves as tools for moving between local and global data. They carry a massive amount of information as they are defined on the open sets of a topological space. If the topology is rich; i.e. there are lots of open sets, then that is a lot of data.

The language of sheaves is indispensable in modern geometry. Also, the language of sheaves is full of agricultural terms such as germ, stalk, and sheaf. Sections of vector bundles are examples of sheaves and so one can to some degree think of a sheaf as sections of a vector bundle.

2 Motivating Example and Definition

We begin with an example. Consider a smooth manifold X. If we have an open set $U \subset X$, we can consider all the smooth functions on U; $C^\infty(U)$ and if $V \subset U$ is an open subset, then we can naturally take an $f_U \in C^\infty(U)$ and restrict it to V: $f_V = f_U|_V$.

Moreover, let's take U_1 and U_2 to be non disjoint open sets such that $U = U_1 \cup U_2$. Now suppose that $f = 0$ when restricted to U_1 and also when restricted to U_2; we expect then that $f = 0$ on U and this is indeed the case for smooth functions. Also, if we have f_1 and f_2 as two smooth functions on U_1 and U_2, respectively, and we know they agree on $U_1 \cap U_2$, then we can **glue** them together to form a function f on U which, when restricted to U_i is f_i.

This example is motivates our definition for a sheaf. But first, a presheaf.

Definition 2.1. *Let X be a topological space. A **presheaf** $\mathcal{F}$ is a contravariant functor from open sets of X to some abelian category A. We may take abelian groups, vector spaces, or R-modules. So $\mathcal{F} : U \mapsto \mathcal{F}(U)$ where $\mathcal{F}(U)$ is some abelian object. Moreover, $\mathcal{F}$ satisfies the following properties.*

1. *If $V \subset U$, then there is a morphism in A $\mathcal{F}(U) \to \mathcal{F}(V)$. It is best to think of this morphism as a restriction map ρ_{UV}. Note the **contravariance**.*

2. *The morphism $\rho_{UU} = \mathrm{id}$ for any open set $U \subset X$.*

3. *If $W \subset V \subset U$, then $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$. This just says that the restrictions are "natural".*

We can now define a sheaf.

Definition 2.2. *A* ***sheaf*** *$\mathcal{F}$ is a presheaf which satisfies two additional properties. Let (U_i) be an open covering of an open set U*

1. *(Locality) If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each U_i, then $s = t$.*
2. *(Gluing) If for each i, an element (also called section) $s_i \in \mathcal{F}(U_i)$ is given such that for each pair U_i, U_j of the covering sets the restrictions of s_i and s_j agree on the overlaps: $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.*

Now, it is also useful to consider local information via stalks.

Definition 2.3. *Let $\mathcal{F}$ be a sheaf on topological space X. A* ***stalk*** *of a point x is denoted $\mathcal{F}_x$ and is simply the direct limit*

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

It may be more useful instead to consider an equivalent definition in terms of ***germs****. Germs correspond to sections over some open set U containing x, and two of these sections are considered the same if they agree on some smaller open set. More precisely: the stalk is $\{(f, U) : x \in U, f \in \mathcal{F}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \in U \cap V$ where $x \in W$ and $f|_W = g|_W$.*

So a stalk is formed from a set of germs. In the case of sections of vector bundles, one might imagine a stalk to be all the sections local to a point x. This "localness" is coming via some sort of a limit. A germ is a particular section in the stalk.

Definition 2.4. *Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. A* ***sheaf morphism*** *$\varphi : \mathcal{F} \to \mathcal{G}$ assigns to each open set U a morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ where this morphism is simply a morphism in the Abelian category.*

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf morphism, we can consider a sheaf $\ker \varphi$ which sends $U \mapsto \ker \varphi_U$. Naively, we might think that we can then define $\operatorname{Im} \varphi$, $\operatorname{coker} \varphi$, or $\mathcal{G}/\mathcal{F}$ similarly to define sheaves. However, in general, they only define presheaves. We're lucky that $\ker \varphi$ automatically defines a sheaf. So we can see that not all presheaves are sheaves. Let's consider another example that doesn't arise from a sheaf morphism.

Example 2.5. let $X = (0, \infty)$ be equipped with the standard subspace topology from $\mathbb{R}$ and L^p be the presheaf which assigns to each open set U the space $L^p(U)$ (equivalence classes of L^p functions). Let $U_n = (0, n)$. Then the "function" $f(x) = x$ restricted to each U_n is in $L^p(U_n)$. However, $X = \bigcup^\infty U_n$ and of course, $f(x) = x$ is not L^p on all of X.

The problem is, in some sense, that some presheaves don't have enough local sections and so they may fail either the locality or gluability conditions (or both). There is, however, a way to turn a presheaf into a sheaf. Suppose $\mathcal{F}$ is a presheaf that is not a sheaf on X. The perspective we take is via the étalé space which is a **covering space** of X (the following works if $\mathcal{F}$ is already a sheaf). So this means $\pi : Et(\mathcal{F}) \to X$ will be a local homeomorphism.

Definition 2.6. *Let $Et(\mathcal{F})$ be, as a set, the disjoint union of stalks:*

$$\coprod_{x \in X} \mathcal{F}_x.$$

There is a natural map $\pi : Et(\mathcal{F}) \to X$ which sends a germ to the point it lies over. Let $x \in U$ and s be a section of $\mathcal{F}(U)$. Define $\bar{s}$ to be a section of $\pi : Et(\mathcal{F}) \to X$ such that x is mapped to the germ s_x. Thus, we give $Et(\mathcal{F})$ the coursest topology such that all these sections $\bar{s}$ are continuous.

Definition 2.7. *The **sheafification** of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ which sends U to the continuous sections of $\pi|_U : Et(\mathcal{F}) \to U$.*

A constant presheaf such as $\mathbb{Z}$ can be sheafified to become a locally constant sheaf still denoted $\mathbb{Z}$. The sheafification of L^p is L^p_{loc}.

3 Sheaf Cohomology

Suppose that we have a short exact sequence of sheaves: $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$.

The functor which takes takes a sheaf to global sections is left-exact but not right-exact. That is, $0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ is a short exact sequence but the morphism $\mathcal{G}(X) \to \mathcal{H}(X)$ in the abelian category is not surjective in general. However, it is often important to know if a section of $\mathcal{H}$ can be lifted to $\mathcal{G}$. Sheaf cohomology, via a resolution, gives us a long exact sequence:

$$... \longrightarrow \mathcal{H} \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow ...$$

Here, the cohomology is defined exactly as you expect: kernel mod image. However, kernel mod image of what sequence? The sequence is some resolution of the original short exact sequence and depending on the type of sheaves, we might be able to resolve with good resolutions. But *a priori*, we may need less tractable resolutions.

3.1 Flasque Sheaves

Thus, our hope is to work with nicer resolutions.

Definition 3.1. *A **flasque sheaf** $\mathcal{F}$ is one such that the restriction morphism $\rho_{XU} : \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for all open sets U.*

In this case, if we have a short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ and $\mathcal{F}$ is flasque, then the global sections **do** form a short exact sequence. Here are two useful lemmas.

Lemma 3.2. *If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is short exact with $\mathcal{F}, \mathcal{G}$ both flasque, then $\mathcal{H}$ is also flasque.*

Lemma 3.3. *Any sheaf $\mathcal{F}$ has a resolution by flasque sheaves:*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \longrightarrow ...$$

Lastly, we have a proposition to show what is the big deal with flasque sheaves:

Proposition 3.4. *If $\mathcal{F}$ is a flasque sheaf, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

3.2 Fine and Soft Sheaves

A few reminders of basic definitions first. An open covering $X = \bigcup_i U_i$ of a topological space is **locally finite** if every point is contained in at most finitely many U_i. A topological space is called **paracompact** if every open cover can be refined to a locally finite open cover. It is not hard to see that a locally compact Hausdorff space with a countable basis is paracompact; e.g. manifolds.

Definition 3.5. *A sheaf $\mathcal{F}$ on a paracompact space X is **fine** if for every locally finite open cover $X = \bigcup_i U_i$, there are sheaf homomorphisms $\eta_i : \mathcal{F} \to \mathcal{F}$, with the following properties:*

1. *There are open sets $V_i \supset X \setminus U_i$, such that $\eta_i : \mathcal{F}_x \to \mathcal{F}_x$ is the zero map for every $x \in V_i$.*

2. *As morphisms of sheaves, $\sum_i \eta_i = \mathrm{id}_{\mathcal{F}}$.*

An example is the sheaf of smooth k-forms on a smooth manifold. Essentially, it is fine because we have partitions of unity.

Definition 3.6. *A sheaf $\mathcal{F}$ on a paracompact topological space is called **soft** if, for every closed subset $Z \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(X)$ is surjective.*

All fine sheaves are soft. Moreover, on a paracompact Hausdorff space, a fine sheaf $\mathcal{F}$ has $H^i(X, \mathcal{F}) = 0$ for $i > 0$. (By the way, if a sheaf has this property of having higher cohomology vanish, it is called **acyclic**). All these ideas can be used to prove the following proposition:

Proposition 3.7. *Let M be a complex manifold and $\mathcal{A}^{p,q}$ be the sheaf of (p,q)-forms. We have $H^i(M, \mathcal{A}^{p,q}) = 0$ for every $i > 0$.*

4 Schemes

We now consider a commutative ring R with 1. In fact, we assume that it is Noetherian: if there is a chain of ascending ideals, it eventually stabilizes. Later, we'll want to consider Noetherian modules which satisfy a similar property but with submodules instead of ideals.

We define the topological space called the spectrum of R:

Definition 4.1. *Let $Spec\, R = \{P \in R : P \text{ is a prime ideal}\}$. The topology is the Zariski topology. Let $I \subset R$ be any ideal in R. Let $V_I = \{P \in Spec\, R : I \subset P\}$. These V_I form the closed sets.*

Spec R is always compact but for most R, not Hausdorff. Its closed points are the maximal ideals so not all points are closed.

Example 4.2. $Spec\, \mathbb{Z} = \{(0), (p) : p \text{ is a positive prime number}\}$. Also, $Spec\, \mathbb{C}[x] = \{(0), (x-a) : a \in \mathbb{C}\}$.

Now let $D(f) = \{P \in Spec\, R : f \notin P\}$. We might think of this as localizing in some sense by formally treating $f \neq 0$ and introducing $1/f$ formally. These $D(f)$ form an open basis of $Spec\, R$. Note that $D(f) \cap D(g) = D(fg)$. We now define a sheaf on $Spec\, R$ that sends $D(f)$ to R_f, the localization of R at f. This is the set $\{r/f^l : r \in R, l \in \mathbb{N}\}$. This is commonly called the **structure sheaf** over $Spec\, R$ and is denoted $\mathcal{O}_{Spec\, R}$. Thus, to $Spec\, R$, we naturally have a sheaf of rings. We call $Spec\, R$ a ringed space for this reason. To be precise, a **ringed space** is a family of (commutative) rings parametrized by open subsets of a topological space together with ring homomorphisms that play roles of restrictions. Moreover, any ringed space isomorphic to something of this form is called an **affine scheme**. Some examples of affine schemes are affine varieties.

But what do we mean by isomorphisms? We mean we get isomorphisms in both the category of rings and also topologically.

We can now state our mantra for projective schemes: **A projective scheme is a locally ringed space which is locally, the spectrum of a ring**. Let's break this down. Firstly, we know that a projective variety is just made by patching together affine varieties. Our case with schemes is motivated by this fact.

A scheme X then, is firstly, a topological space. Thus, locally, we can talk about homeomorphisms with the underlying topology. Secondly, a scheme is not only a topological space but comes equipped with a sheaf. This large sheaf can be viewed as sitting over X and if $U_i \subset X$

are open sets, then the sheaf considered over U_i are some $Spec\, R_i$. Note that thus, locally, a projective scheme is an affine scheme! So we now have that complex manifolds are examples of schemes. (by GAGA?)

Alright, we're badly need of an example.

Example 4.3. Consider the rings $\mathbb{C}[x]$ and $\mathbb{C}[y]$. The spectra of these are both basically $\mathbb{C}^*$. If we localize at x and y, we get the rings $\mathbb{C}[x, x^{-1}]$ and $\mathbb{C}[y, y^{-1}]$. Now, if we take the map $x \mapsto y^{-1}$, this gives us a way to glue these two affine pieces together to obtain $\mathbb{C}P^1$. The punctures in each tells us precisely that each chart covers everything except for one point on $\mathbb{C}P^1$.

Remark: One may ask, "What is the point of a scheme, anyways?" One answer I've had which may or may not be helpful is that we know classically that Hilbert's Nullstellensatz gives us a 1-1 correspondence of affine varieties and radical ideals of a polynomial ring over an algebraically closed field. But is there a way to expand this beyond radical ideals.

5 Coherent Sheaves

The story of coherent sheaves is not much of an extension from sheaves. Instead of Noetherian rings, we move to Noetherian modules which have the property: if M is a finitely generated module, then all its submodules are finitely generated. Let R be a Noetherian ring and M an R-module. As above, we can define a sheaf $\tilde{M}$ that sends the open set $D(f)$ in $Spec\, R$ to $M_f := \{m/f^l : m \in M, l \in \mathbb{N}\}$.

The coherence condition is that the modules should all be finitely-generated. If these were quasi-coherent sheaves, then they don't need to be finitely-generated.

Example 5.1. Here are some coherent sheaves:

1. Structure sheaf $\mathcal{O}_{Spec\, R}$
2. Skyscraper sheaves S; these are defined thusly. Fix an $x \in X$ and let M be a finitely generated module. Let U be an open set
$$S(U) = \begin{cases} M, & x \in U \\ \{0\}, & x \neq U \end{cases}.$$
Since in the Zariski topology, there are points which aren't closed, they kind of "spread out" like butter. And so skyscraper sheaves can also bs spread out; nearby points to a non-closed point x have the same picture.
3. Ideal Sheaves: Let Y be a closed scheme of a scheme X. We get a surjective morphism $\varphi : \mathcal{O}_X \to i_*\mathcal{O}_Y$ where i_* is induced by inclusion. The ideal sheaf is $\ker \varphi$.
4. Locally Free Sheaves of finite rank: Let X be a scheme. These are very much like vector bundles. Locally, it looks like $\mathcal{O}_X \oplus ... \oplus \mathcal{O}_X$ finitely many times. Well, the structure sheaf $\mathcal{O}_X$ is a trivial bundle and so locally, we have a trivialization which is of course, what we require of vector bundles.

6 A Few Theorems

Theorem 6.1. *Let X be a projective scheme over a Noetherian ring and $\mathcal{T}$ be a coherent sheaf over X. Then*

1. $H^i(X, \mathcal{T})$, $i > 0$, *is a finitely generated* R*-module.*

2. $H^i(X, \mathcal{T}(n)) = 0$, $i > 0$, *for large* $n > n_0$. *This* $\mathcal{T}(n) = \mathcal{T} \oplus \mathcal{O}(n)$ *for example, on* $\mathbb{C}P^n$.

Theorem 6.2. *Let* $\mathcal{F}$ *be a coherent sheaf over a smooth projective variety* X. *Then* $\mathcal{F}$ *admits a resolution by vector bundles of finite length:*

$$0 \longrightarrow \mathcal{E}^n \longrightarrow ... \longrightarrow \mathcal{E}^1 \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

In fact, $n = \dim X$.

Theorem 6.3. *Let* X *be a smooth projective variety and suppose the anticanonical line bundle* ω_X^* *is ample. Then* $DCoh(X)$ *determines* X *(the derived category of coherent sheaves).*