HMS for $\mathbb{P}^1$

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1 Introduction

Let X be a compact Kähler and D a reduced simple normal crossing divisor such that $D \in |-K_X|$. Reduced means the coefficients are $+1$ which makes it an effective divisor. Simple normal crossing just means the crossing is like that of hyperplanes intersecting. And the last condition is that D is of anticanonical class. For example, if $X = \mathbb{P}^1$, then the anticanonical class is $\mathcal{O}(2)$ which is just two points; we can identify them with 0 and ∞.

Let $X^0 = X - D$. This is like an open Calabi-Yau in that it has a holomorphic volume form; on X, the poles would lie in D. Suppose now that we know the mirror of X^0; call it $\check{X}$. How do we find the mirror for the original X? We consider something called a Landau-Ginzburg model: $(\check{X}, W)$ Here, $W : \check{X} \to \mathbb{C}$ is a holomorphic function. Physicists call it a **superpotential**.

Fact that was well-known to physicists a long time ago: Let X be a toric, Fano variety. This means it contains, as an open dense set, $(\mathbb{C}^*)^n$ and has ample anticanonical bundle. Let $D = X - (\mathbb{C}^*)^n$; then there is an explicit mirror for $X^0 : X - D$ which is just $\check{X} = (\mathbb{C}^*)^n$ and $W = \sum(\text{monomials})$. The monomials are in 1-1 correspondence with the facets of the defining polytope of the toric variety. All $\mathbb{P}^n$ are toric and Fano.

The polytope for $\mathbb{P}^1$ is just a line with some marked point (I don't really understand what it means but apparently the two components of the line correspond to the two charts for defining $\mathbb{P}^1$). Here, $W = z + 1/z$.

Now, $\mathbb{P}^1$ is Kähler so we can view it as a symplectic manifold or as a complex manifold. When viewing it as a symplectic manifold, I'll denote it as S^2 instead. We have two mirror symmetry statements and thus, four categories.

1. $\mathbb{P}^1$ is mirror to $(\mathbb{C}^*, W = z + 1/z)$. That is, $D^b_{coh}(\mathbb{P}^1) \cong D^\pi(\mathcal{FS}(W))$; the RHS has the Fukaya-Seidel category which we'll define.

2. S^2 is mirror to $(\mathbb{C}^*, W = z + 1/z)$ and the categories are $D^\pi(\mathcal{F}(S^2)) \cong D^b_{sing}(W)$.

2 First Statement

2.1 B-Side: Complex Geometry

In D^b_{coh}, an exceptional object A is such that

$$\hom(A, A[p]) = \begin{cases} 0, & p \neq 0 \\ \mathbb{C}, & p = 0. \end{cases}$$

A strong exceptional sequence is a collection $A_0, ...A_n$ of exceptional objects such that there are morphisms from $A_i \to A_j$ when $i < j$ but the only morphisms from $A_j \to A_i$ are trivial. It is a theorem of Grothendieck that any holomorphic vector bundle on $\mathbb{P}^1$ splits (not true even for $\mathbb{P}^2$). There is a theorem by Beilinson: $D^b_{coh}(\mathbb{P}^n)$ is generated by a strong exceptional sequence $\langle \mathcal{O}, \mathcal{O}(1), ..., \mathcal{O}(n)\rangle$. In the case of $\mathbb{P}^1$, the proof is simplified using Grothendieck's result. Note: tensoring by line bundles preserves short exact sequences.

However, more generally, if D^b_{coh} is generated by, say, $F_1, ..., F_n$, then this implies that $K^0_{alg} \cong K_0^{alg}$ (K-theories; don't ask me what this means). This means that $K^0_{alg} \otimes \mathbb{C} \cong \mathbb{C}\langle [F_1], ..., [F_n]\rangle$, a finite dimensional vector space. However, $K^0_{alg} \otimes \mathbb{C} \cong CH^*)X_{\mathbb{C}}$, the Chow group. This is typically an enormous group and so, saying that D^b_{coh} is generated in this manner is an extremely strong condition to impose. The Hodge diamond of $\mathbb{P}^n$ is nonzero only on the vertical so we can recover the cohomology from the Chow group.

Anyways, we have then that $D^b_{coh}(\mathbb{P}^1) \cong \langle \mathcal{O}, \mathcal{O}(1)\rangle$. There is some quiver representation to show that this is actually $Mod(\mathbb{C}Q)$ where $\mathbb{C}Q$ is the path algebra over $\mathbb{C}$ for the quiver representation of $\mathbb{P}^1$. Whatever all that means, the important bit is the following:

- $\hom(\mathcal{O}, \mathcal{O}) = \mathbb{C}$.
- $\hom(\mathcal{O}, \mathcal{O}(1)) = \mathbb{C}^2$
- $\hom(\mathcal{O}(1), \mathcal{O}(1)) = \mathbb{C}$
- $\hom(\mathcal{O}(1), \mathcal{O}) = 0$ by the exceptional condition.

2.2 A-Side: Symplectic Geometry

Let's look at the symplectic mirror now: $(\mathbb{C}^*, W = z + 1/z)$. Let's define, in generality first, what the Fukaya-Seidel category of a Landau-Ginzburg model $(\check{X}, W)$ is.

The objects of $\mathcal{FS}(W)$ are **admissible** Lagrangians. Let L be a Lagrangian. It is admissible if there is a compact set $K \subset \check{X}$ such that $W(L - K)$ is a path in $\mathbb{C}$ which approaches the real axis Re_+. I think it must do so asymptotically so possibly, we cannot have the path cross Re_+ infinitely many times. The morphisms are as before: $\mathbb{C}$ linear combinations of intersection points of two admissible Lagrangians.

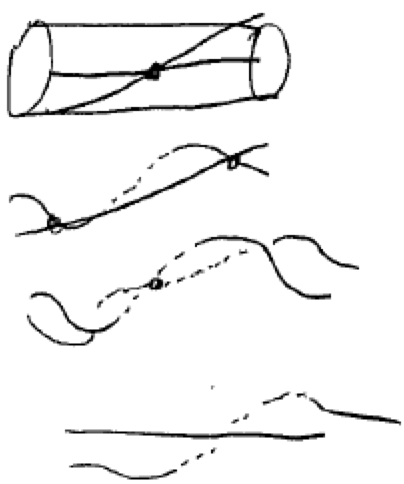
Looking at $z+1/z$, this goes to $+\infty$ on Re_+ if and only if $z \to 0$ or ∞. Thus, in $\mathcal{FS}(z+1/z)$, the only interesting admissible Lagrangians are L_k where L is a cotangent fiber (we identify $\mathbb{C}^* \cong T^*S^1$) and L_k is L wrapped around the cylinder k times. Allowing for Hamiltonian isotopy, we have the following intersections for L_0 and L_1.

$\mathrm{Hom}(L_0, L_0) = \mathbb{C}$

$\mathrm{Hom}(L_0, L_1) = \mathbb{C}^2$

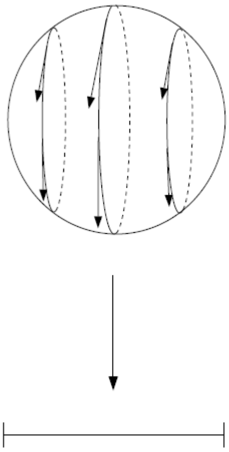
$\mathrm{Hom}(L_1, L_1) = \mathbb{C}$

$\mathrm{Hom}(L_1, L_0) = 0$



We observe this is similar to what we had above with the morphisms between $\mathcal{O}$ and $\mathcal{O}(1)$. Thus, we've shown the mirror symmetry for statement 1.

Perhaps another way to look at this is to consider $\mathbb{P}^1$ and its moment map which projects it down to the real line. The fibers here are S^1's except at the ends where they degenerate to points.



Then, in the dual fibration, we have something like a cylinder. One should think of the great circle on $\mathbb{P}^1$ as corresponding to the skinniest part of the cylinder. As we move towards the two points where the circles vanish, this corresponds to larger and larger circles on the mirror. And sort of the point is that they are infinitely large at the end. If we try to wrap the Lagrangians at the extreme, it will be like trying to travel infinite distance. Hence, we only partially wrap.

3 The Second Statement

Yoonjoo did not talk much about the symplectic side. We note that somehow, the only interesting Lagrangian that we can work with is a great circle which admits two local systems. So we have two objects.

Here is a theorem by Serre. Let Y be a complex variety. Then Y is smooth if and only if there exists $N \gg 0$ such that $pd(F) \leq N$ for all coherent sheaves F on Y. Here, $pd(F) =$ smallest length of locally free resolutions of F; it is called the projective dimension. This gives a way to measure how singular Y is. Let $Perf(Y)$ be objects of D^b_{coh} that are quasi-isomorphic to complexes of locally free sheaves. When Y is smooth, we have that $Perf(Y) = D^b_{coh}$ and thus, $D^b_{coh}(Y)/Perf(Y) = 0$. But in general, this is not 0.

We let

$$D^b_{sing}(W) = \prod_{\lambda \in \mathbb{C}} D^b(\check{X}_\lambda)/Perf(X_\lambda)$$

where $\check{X}_\lambda$ is the fiber $W^{-1}(\lambda)$.

For us, W is a degree 2 map since if $W = \lambda$, we get a quadratic $z^2 - \lambda z + 1$. The critical points of W are found by $W'(z) = 1 - 1/z^2 = 0$: $z = \pm 1$. Then, when $\lambda = \pm 2$, we have $z + 1/z = \pm 2 \iff (z \pm 1)^2 = 0$. So letting $Z = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$, $D^b_{sing}(W) = (D^b(Z)/Perf(Z))^{\times 2}$. It turns out this latter thing is isomorphic to $D^b(\mathrm{Mod}(C))^{\times 2}$. Here $C = \mathbb{C}[x]/(x^2 - 1) \cong \mathbb{C} \oplus \mathbb{C}j \cong \mathbb{H}$ (quaternions). It may be that in general, we get Clifford algebras appearing, like this one.

So somehow, we need to realize that $D^\pi(\mathcal{F}(S^2))$ is equivalent to this category. This may be discussed in a future talk.