Notes on *A Beginner's Guide to the Fukaya Category*

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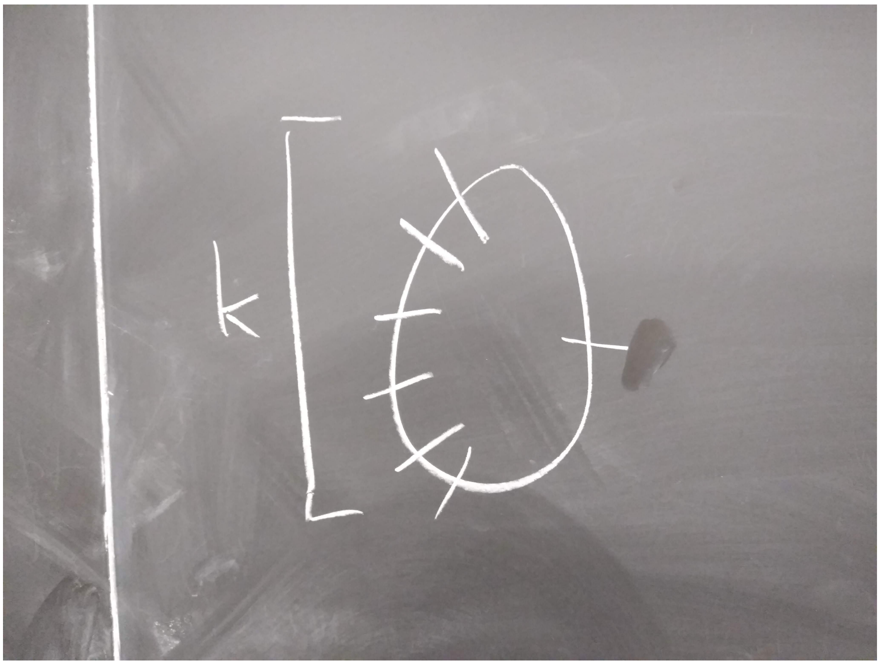
These are some notes I'm compiling for the Fall 2019 RTG seminar which is on homological mirror symmetry. One of the main resources is Denis Auroux's *A Beginner's Guide to the Fukaya Category*. Much of these notes come from discussions with Andrew Hanlon and Catherine Canizzo. Both of them were students of Denis Auroux.

1 Moduli Space of Holomorphic Disks with Marked Points

Let $\mathcal{M}_k$ denote the space of holomorphic disks with $(k+1)$ marked points, up to automorphisms. We have that $\dim \mathcal{M}_k = (k+1) - 3 = k - 2$. The reason for writing it this way is to recall that $\operatorname{Aut}(D^2) = PSL(2, \mathbb{R})$ which is 3 dimensional.



In the Fuakaya category, the k points on the left are mapped to the right point

Claim: $\mathcal{M}_k$ has a stable disk compactification $\overline{\mathcal{M}}_k$. If we have a sequence u_n in $\mathcal{M}_k$, we'll like it to admit a subsequence with some limit. The limits arise from two phenomena:

1. Strip Breaking

2. Degeneration of Domain

The reason we can compactify the first type is basically because of Gromov compactness. I'm not really sure how this works. But we have the phenomenon of disk bubbling and I think an explanation of how Gromov compactness works with dusk bubbling can help explain strip breaking.

So we basically need to show that a sequence u_n has a subsequence converging to something with **finitely** many bubbles. The energy of u is defined

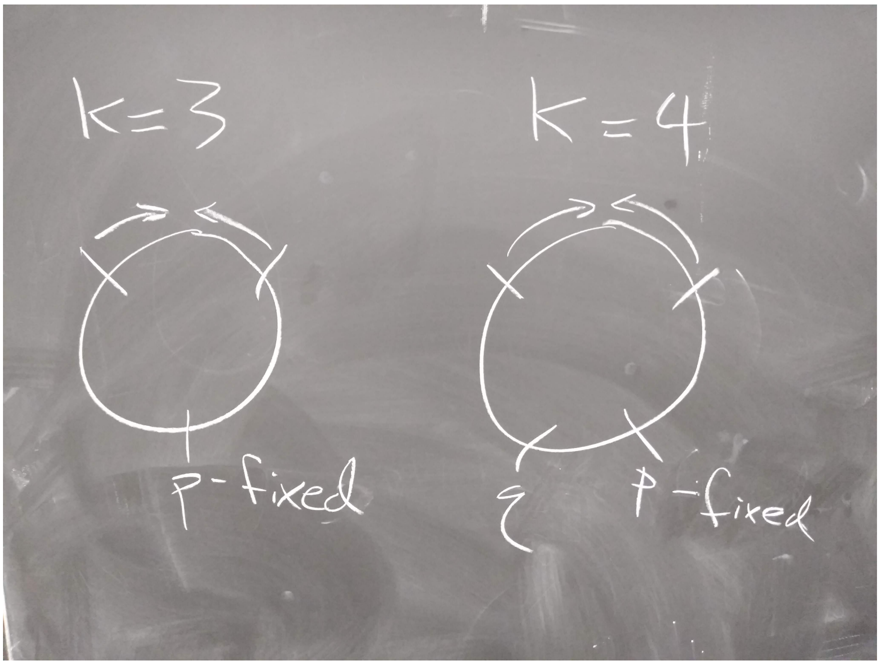
$$E(u) = \int_D |du|^2 = \int_D \left|\frac{\partial u}{\partial s}\right|^2 = \int_D u^*\omega.$$

The second equality follows by J-holomorphicity. When we consider things with bounded energy, note that it is still possible to have $|du|$ grow to infinity if the compact support of du shrinks to a point. And this is how bubbles form. However, since the energy is finite, then the number of bubbles we get has to have finite collective energy. Okay, but it's possible to have infinitely many bubbles with their energy decaying proportional to, say, $1/n^2$.

We rule this out because $u^*\omega \in H^2(D, \mathbb{Z})$ and so the possible areas it has are discrete. This gives a lower bound on what the energy of the disks can be. Having both a lower and upper bound, there has to be only finitely many. This is rather interesting; it's sort of saying the energy of the bubbles is **quantized**. It's not so surprising to learn that J-holomorphic curves appear in string theory on the quantum side.

I believe the reasoning above works for strip breaking as well.

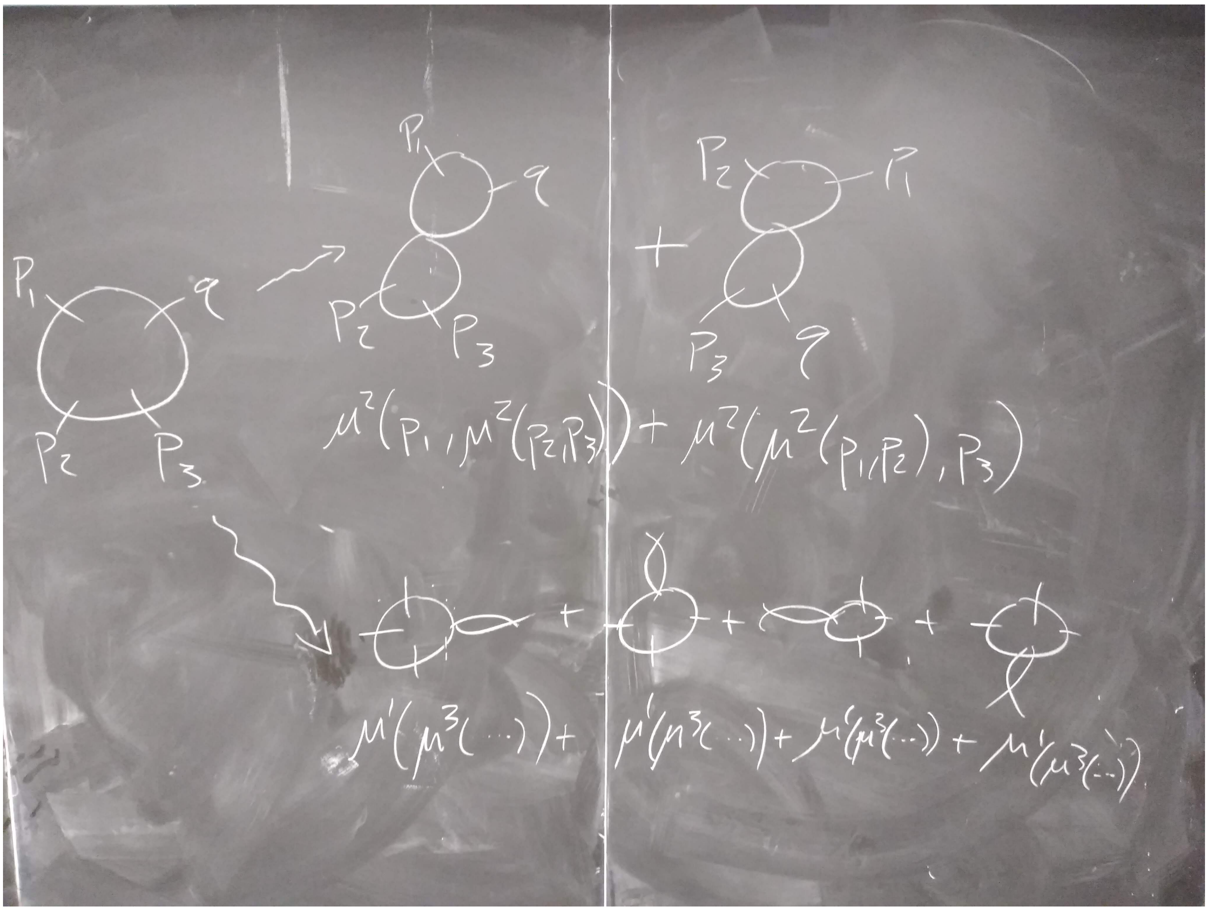
As for the second type, degeneration of domain, the idea is that when we have more than 3 marked points, we can have some interesting behavior. Consider the following picture.



On the left, when $k = 3$ and p is a fixed point, we can use an automorphism to separate the two points that get close to each other. But if $k = 4$ or anything higher, when the two points get close and p is fixed, then trying to separate them with an automorphism will force q to move towards p. And thus, in the limit, separating a pair amounts to making another pair collide.

2 μ^3 and Associativity

We can define higher order operations using the geometry. Let's just focus on μ^3 because it shows that the Floer product, μ^2, is associative only up to homotopy. Let $k = 3$; so we're looking at $\mathcal{M}_3$. Consider the following picture. The left most disk is marked with 4 points and the bottom row corresponds to strip breaking while the top row corresponds to degeneration of domain.



μ^3

In general, if Ind $[u] = k$, then when we have this sort of breaking or degeneration, the sum of the indices of all the disks/strips has to add up to k. We'll explain this in a moment in our case where $k = 3$. If we fix our points p_1, p_2, p_3, then $\mathcal{M}(p_1, p_2, p_3, [u], J)$ where $[u]$ is some homotopy class of J-holomorphic disk. The dimension of this space is Ind $[u] + k - 2$. Since we want the dimension to be 1 so that we can study compact 1-manifolds with boundary and count boundary points. Anyways, the point is that then when $k = 3$, Ind $[u] = 0$.

The explanation for why the index remains unchanged despite breaks and degeneration is that the moduli space comes from studying some Fredholm operator and solutions to some PDE defined by it. The Fredholm index is a topologial sort of thing (as demonstrated by index theory) and so the index should not change under these sorts of breakings and degenerations. So in the strip breakings, we get the indices become $+1$ on one piece and -1 on the other, summing to 0. In the degenerations, they both are 0.

Let's explain the top row of the picture. We want to obtain q at the end. How can do we do that with the points p_1, p_2, p_3? If we take $\mu^2(p_2, p_3)$, this gives us the node point connecting the two disks. Call it r. Then $\mu^2(p_1, r)$ gives the final point which is q. A similar story is told for the other term.

The bottom row has four terms which considers μ^3 on three points which is a node. Then μ^1, which is the differential, sends the node to the last point, at the other end of the strip.

Now, on the homology level, the terms we see in the bottom row all vanish. Thus, on the homology level, we have $\mu^2(p_1, \mu^2(p_2, p_3)) = \mu^2(\mu^2(p_1, p_2), p_3)$. Written another way makes associativity clear: $p_1 \cdot (p_2 \cdot p_3) = (p_1 \cdot p_2) \cdot p_3$.