AIM PROBLEM LIST
DAVID STAPLETON

Question 1. Suppose $A$ is an ample (or nef) vector bundle. Is each $c_kA$ pseudo-effective? (Fulton-Lazarsfeld showed they are nef, and the answer is yes if $E$ is globally generated.)

(Ex. If $X$ is an abelian variety)

Question 2. Are Schur polynomials in the Chern classes pseudo-effective?

Reminder: Fixing a partition $\lambda := (\lambda_1 > ... > \lambda_r)$, then the Schur polynomial is the determinant of the matrix

$$
\begin{vmatrix}
    c_{\lambda_1} & c_{\lambda_1 + 1} & \cdots \\
    c_{\lambda_1 - 1} & c_{\lambda_2} & \cdots \\
    \vdots & \vdots & \ddots
\end{vmatrix}
$$

Question 3. Do Schur polynomials of nef vector bundles generate $\text{Nef}^k(X)$?

Question 4. Suppose $P_1$ and $P_2$ are Schur polynomials in nef vector bundles. Is $P_1 \cdot P_2$ nef? (Note: the product of two Schur polynomials is a positive sum of Schur polynomials, but we could take the polynomials in different vector bundles)

Example 1. Try to compute $\text{Eff}_k(X)$ on high degree very general hypersurfaces in $Gr(k, n)$ or $\mathbb{P}^r \times \mathbb{P}^m$?

Question 5. How many very general points must you blow-up in $Gr(k, n)$ before it ceases to be a Mori Dream Space? (the $\mathbb{P}^n$ case is known) but the proof involves existence of a big "Cremona" action.

Question 6. For what classes of varieties can we guarantee equality of various "dual positive cones"? (E.g.s Nef, Upsef, BPF. and pliant)

Definitions: The universal pseudo-effective cone $\text{Upsef}^k$ consists of classes $\beta \in N^k(X)$ such that for all $f : Y \to X$, $f^*(\beta) \cap [Y] \in \text{Eff}_*(Y)$. ($\text{Upsef} \implies \text{Nef}$)

The base point free cone consists of products of basepoint free classes.

The pliant cone consists of products of pullbacks of Schubert classes from Grassmannians (=products of Schur polynomials in globally generated vector bundles).

Question 7. For what kinds of varieties can one guarantee equality of various "dual positive" cones? (E.g. Nef $\supset$ Upsef $\supset$ BPF $\supset$ pliant)

Question 8. Is $\text{Upsef}=\text{Nef} \cap \text{Eff}$? (true for spherical/toric/homogeneous varieties)
Question 9. Do duals of these cones have interesting geometric interpretations?

Question 10. If $\beta \in N^k(X)$ and for all inclusions $i : Y \hookrightarrow X$, $i^*\beta \cap [Y] \in \text{Eff}_i(Y)$, does this imply $\beta \in U_{pse} f^*$?

Example. Compute $\text{Eff}_k$ on projective bundles in the following cases. (Fulger computed these for $\mathbb{P}^r$-bundles over curves. Everything falls out of the Harder-Narasimhan filtration)
1) Split bundles.
2) $\mathbb{P}^2$-bundles over $\mathbb{P}^2$ or a K3 surface.
   • Are there non-finitely generated cones?, • non-simplicial cones?, test the strong conjecture of Debarre-J-Voisin. (The weak conjecture is known in this case.)

The Strong and Weak Conjecture. Suppose $f : X \rightarrow Y$ is a morphism. Let’s look at contracted classes in $\text{Eff}_k(X)$:

$$\text{Eff}_k(X) := \{ \alpha \in \text{Eff} | f_*\alpha = 0 \}$$

$$= \{ \alpha \in \text{Eff} | f^*H^k \cdot \alpha = 0, H \text{ ample} \}.$$

• Strong Conjecture: Is $\text{Eff}$ the closure of the cone generated by contracted effective classes?
• Weak Conjecture: Is $\text{Eff}$ in the vector space spanned by effective contracted classes?

Example. Let $X$ be a projective variety. If we blow-up $X$ at enough very general points. Is it always true that each $\text{Eff}$ is not finitely generated?

Example. $\text{Eff}(X)$ for $X = \text{Hilb}^n(S)$? $S = \mathbb{P}^2$?

Example. $\text{Eff}_2(X)$ for $X = M^{En}_0$?

Example. $\text{Eff}$ for blowup $\mathbb{P}^n$ over subvarieties of dimension $\geq 1$?

Question 11. Does flat pullback of cycles descend to numerical equivalence?

Question 12. If $X$ is irreducible and non-reduced, is $N_k(X_{\text{red}}) \rightarrow N_k(X)$ an isomorphism?

Question 13. For $X$ singular, do $k$-th Chern classes span $N^k(X)$?

Question 14. How do numerical spaces behave in smooth families? Is the dimension constant for the very general member?

Example. Is $\text{Eff}_2(X)$ finitely generated for a 2-Fano variety $X$? (Defn of 2-Fano: $X$ Fano, and $c_{h^2}(\Omega_X)$ nef?)

Question 15. Can we relate positivity of the Chern classes of $\Omega_X$, $T_X$ to the geometry of $X$?

Question 16. Are there characterizations of positive cones of vector bundles in terms of metrics? Or positive cones of subvarieties in terms of currents?

Example. Is the diagonal on $S \times S$ rigid where $S$ is a smooth surface of general type, $q(S) = 0$ and $g(S) \geq 0$? ($Z$ rigid if it the unique effective $\mathbb{R}$-cycle in its numerical class) What about $S$ a K3? (it is known for very general K3 surfaces by work in progress by Ottem and Lehman) Or positivity of
Question 17. Given a rational contraction $f : X \to Y$ with $\dim(Y) < \dim(X)$, such that 2 distinct irreducible divisors $D_1$ and $D_2$ satisfy $f(D_1) = f(D_2) \neq Y$, then $D_1$ and $D_2$ are extremal. [Dave Jensen’s genus 5 and genus 6 paper] Is there an analogue in higher codimension?

Question 18. Given a curve $C \subset X$ is a curve with ample normal bundle, does some multiple of $C$ deform? (i.e. an algebraic family of irreducible curves with each fiber equivalent to $m[C]$, it is known that $[C]$ is big [Ottem]. It is false if $C$ is a surface [Fulton-Lazarsfeld].)

Question 19. What is the cone of ample subvarieties (in Ottem’s sense—see Ottem’s thesis)?

Question 20. If $V, W \subset X$ have complementary codimension and ample normal bundles, is $V \cap W \neq \emptyset$?

Question 21. Is there an analogue of Kawamata-Morrison cone conjecture for abelian 4-folds?

Question 22. Can we compute $\text{Eff}_2$ of $S \times S$ or $\text{Hilb}^2S$ for $S$ any K3 surface? (Note: the Hodge conjecture is not necessarily known in this case.) What about $\text{Sym}^2 C$?

Definition: Given $\alpha \in N_k(X)_Z$, the mobility count

$$mc(\alpha) := \max \text{number of general points we can impose on an effective cycle of class } \alpha.$$  

This is an analogue of $\dim H^0(X, L)$. The expected growth rate: $mc(m\alpha) \approx Cm^{n/(n-k)}$.

Goal: Understand behavior of $mc(m\alpha)$ as $m \to \mathbb{Z}$.

The mobility is

$$\text{mob}(\alpha) := \frac{mc(m\alpha)}{m^{n/(n-k)} / n!}.$$  

The Iitaka dimension

$$K(\alpha) = \max \{ r \in \mathbb{R} | \limsup_{m \to \infty} \frac{mc(m\alpha)}{m^r} \}.$$  

Question 23. (Hard) Compute $\text{mob}(\alpha)$ for $\alpha =$ line class on $\mathbb{P}^3$? (slogan: “complete intersections are optimal”)

Question 24. What is $K(\alpha)$ for $\alpha$ a Schubert class on $Gr(k, n)$?

Question 25. Develop better estimates for the growth rate of $mc(m\alpha)$ for $\alpha$ on $Gr(2, 4)$ or $\mathbb{P}^2 \times \mathbb{P}^2$.

Question 26. Is $K(\alpha) \in \mathbb{Z}$? (Yes for divisors, curves, $Gr(2, n)$, probably $\mathbb{P}^r \times \mathbb{P}^k$, etc...) What is the geometric meaning?
Question 27. (Voisin) Suppose $\alpha$ is represented by a family of cycles whose tangent spaces at a general point attain "all possible directions". Is $\alpha \in \text{Eff}_k$?

Question 28. Is there an Okounkov body construction for curves?... in threefolds? (the volume of a curve is defined to be

$$\text{vol}(C) := \inf_{A \text{ ample}} \left( \frac{A \cdot C}{\text{vol}(A)^{1/n}} \right)^{n/(n-1)}$$

this is a "polar transform" of the standard volume function for divisors.

Question 29. For a toric variety, find a convex body interpretation of the volume for curves (or Zariski decomposition)? ... or for vector bundles?