

A note on the Intersection of Veronese Surfaces

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0. Introduction

The main purpose of this note is to prove the following

Theorem 0.1 *Any two Veronese surfaces in \mathbb{P}^5 whose intersection is zero-dimensional meet in at most 10 points (counted with multiplicity).*

Our initial motivation for this note comes from our paper [EGHPO] where we study linear syzygies of homogeneous ideals generated by quadrics and their restriction to subvarieties of the ambient projective space with known (linear) minimal free resolution. A direct application of the techniques in [EGHPO, Section 3] shows that the homogeneous ideal of a zero-dimensional intersection of two Veronese surfaces in \mathbb{P}^5 is 5-regular (see also Lemma 1.1 below), which yields only an upper bound of 12 for its degree, cf. Section 1.

Section 2 analyzes Veronese surfaces on hyperquadrics. The observation that two Veronese surfaces on a smooth hyperquadric $Q \subset \mathbb{P}^5$ meeting in a zero-dimensional subscheme, must meet in a subscheme of length 10 or 6 is classical and goes back to Kummer [Ku] and Reye [Rey] (see also [Jes1] for historical comments): By regarding the smooth hyperquadric $Q \subset \mathbb{P}^5$ as the Plücker embedding of the Grassmannian of lines $\text{Gr}(\mathbb{P}^1, \mathbb{P}^3)$, a Veronese surface on $Q \subset \mathbb{P}^5$ is, up to duality, the congruence of secant lines to a twisted cubic curve and thus has bidegree $(1, 3)$. More precisely, the congruence has one line passing through a generic point of \mathbb{P}^3 and 3 lines contained in a generic plane. Thus Schubert calculus yields that the possible intersection numbers of two Veronese surfaces on Q are either 10 or 6. See Proposition 2.1 below and the following remark, or the computation of the number of common chords of two space curves in [GH, page 297].

The case of two Veronese surfaces in \mathbb{P}^5 meeting in 10 simple points has also been investigated in relation with association (projective Gale transform) by Coble [Cob], Conner [Con] and others. In [Cob, Theorem 26] Coble claims that 10 points in \mathbb{P}^5 which are associated to the 10 nodes of a symmetroid in \mathbb{P}^3 , the quartic surface defined by the determinant of a symmetric 4×4 matrix with linear entries in \mathbb{P}^3 , are the (simple) intersection points of two Veronese surfaces in \mathbb{P}^5 (see Proposition 3.3 below). This is based on Rey's observation [Rey, page 78-79] that 4×4 symmetric matrices with linear entries in \mathbb{P}^3 are actually catalecticant with respect to suitable bases and on the analysis in [EiPo] of the Gale Transform of zero-dimensional determinantal schemes. Section 3 contains a modern account of these results.

In Section 4 we briefly discuss which intersection numbers ≤ 10 can actually occur for two Veronese surfaces in \mathbb{P}^5 and in which geometric situation this can happen. For instance, we show that two Veronese surfaces in \mathbb{P}^5 cannot intersect transversally in 9 points, however they may intersect in non-reduced zero-dimensional schemes of this degree.

With the exception of Section 3 all other results in this note are valid in arbitrary characteristic.

1. A reduction step

We shall make essential use of the following lemma whose proof is reminiscent of the linear syzygies techniques used in [EGHPO, Section 3].

Lemma 1.1 *If X_1 and X_2 are two Veronese surfaces in \mathbb{P}^5 meeting in a zero-dimensional scheme W , then the ideal sheaf $\mathcal{I}_{W, \mathbb{P}^2}$ of W regarded as a subscheme of \mathbb{P}^2 is 5-regular.*

Proof. The claim is equivalent to the vanishing $h^1(\mathcal{I}_{W, \mathbb{P}^2}(4)) = 0$. In order to see this we consider the minimal resolution of \mathcal{I}_{X_1} in \mathbb{P}^5 and restrict it to X_2 . This yields a complex abutting to \mathcal{I}_W . Since X_1 has property N_p for all p one immediately computes that $h^1(\mathcal{I}_{W, \mathbb{P}^2}(4)) = 0$. \square

As immediate consequence we get a first bound for the number of points where two Veronese surfaces whose intersection is zero-dimensional can meet.

Proposition 1.2 *Two Veronese surfaces in \mathbb{P}^5 whose intersection is zero-dimensional meet in at most 12 points.*

Proof. Let X_1 and X_2 be two Veronese surfaces meeting in a zero-dimensional scheme W , and let $d = \text{length}(W)$. By Lemma 1.1, W regarded as a subscheme of $X_2 \cong \mathbb{P}^2$ imposes independent conditions on plane quartics, in particular if $d \leq 15$. On the

other hand $X_2 \subset \mathbb{P}^5$ is cut out by quadrics scheme-theoretically, and thus the quartics in $H^0(\mathcal{I}_{W, \mathbb{P}^2}(4))$ must cut out $W \subset \mathbb{P}^2$ scheme-theoretically too – in particular there are at least two. If there were only two, then they would form a complete intersection, generating a saturated ideal, and thus $W \subset \mathbb{P}^2$ would be a complete intersection of two plane quartics, which cannot be 5-regular. It follows that $h^0(\mathcal{I}_{W, \mathbb{P}^2}(4)) \geq 3$, and thus that $d \leq 12$. \square

Actually the above proof yields also the following estimate

Proposition 1.3 *Let X_1 and X_2 be two Veronese surfaces in \mathbb{P}^5 meeting in a zero-dimensional scheme W of length d with $10 \leq d \leq 12$. Then $X_1 \cup X_2$ lies on at least $d - 9$ quadrics.*

Proof. Let $a = h^0(\mathcal{I}_{X_1 \cup X_2}(2))$. Then on one hand $h^0(\mathcal{I}_{W, \mathbb{P}^2}(4)) \geq h^0(\mathcal{I}_{X_2}(2)) - a = 6 - a$, on the other hand, by Lemma 1.1, we know that $h^0(\mathcal{I}_{W, \mathbb{P}^2}(4)) = 15 - d$. Combining the two proves the claim of the proposition. \square

2. Veronese surfaces on hyperquadrics

In this section we analyze the intersection of two Veronese surfaces which meet in finitely many points, in the case where the two surfaces lie on a common hyperquadric, resp. a pencil of hyperquadrics. We begin with the classical case of congruences of lines:

Proposition 2.1 *Assume that X_1 and X_2 are Veronese surfaces which meet in finitely many points and assume moreover that there exists a smooth quadric hypersurface Q with $X_1 \cup X_2 \subset Q$. Then $X_1.X_2 = 6$ or 10 .*

Proof. Since Q is smooth it is isomorphic to the Grassmannian $\text{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ and it is well known that $H^4(\text{Gr}(\mathbb{P}^1, \mathbb{P}^3), \mathbb{Z}) = \mathbb{Z}\alpha + \mathbb{Z}\beta$ where α and β are 2-planes. It follows from the double point formula (see for instance [HS]) that every Veronese surface on $\text{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ has class $3\alpha + \beta$ or $\alpha + 3\beta$. Since $\alpha^2 = \beta^2 = 1$ and $\alpha\beta = 0$ it follows that $X_1.X_2 = 10$ or 6 depending on whether X_1 and X_2 belong to the same class or not. \square

Remark As already mentioned in the introduction, a Veronese surface of class $3\alpha + \beta$ on the Grassmannian $\text{Gr}(\mathbb{P}^1, \mathbb{P}^3) \cong Q \subset \mathbb{P}^5$ is the congruence of secant lines to a twisted cubic curve (where β is the cycle of lines passing through a point of \mathbb{P}^3 while α is the cycle of lines in a plane). Passing to the dual \mathbb{P}^3 exchanges α and β , so up to duality the same construction accounts also for Veronese surfaces of class $\alpha + 3\beta$. It is easy to see, for instance by using Kleiman's transversality theorem, that both cases described in Proposition 2.1 actually occur.

Proposition 2.2 *Assume that X_1 and X_2 are Veronese surfaces in \mathbb{P}^5 which meet in finitely many points and assume that there exists a rank 5 hyperquadric Q containing both X_1 and X_2 . Then $X_1.X_2 = 8$.*

Proof. We first claim that X_1 and X_2 do not pass through the vertex P of the quadric cone Q . Otherwise projection from P would map the Veronese surface to a cubic scroll contained in a smooth hyperquadric $Q' \subset \mathbb{P}^4$. But this is impossible, since by the Lefschetz theorem every surface on Q' is a complete intersection and hence has even degree. Blowing up the point P we obtain a diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{p} & Q \\ \downarrow \pi & & \\ Q' & & \end{array}$$

where π gives \tilde{Q} the structure of a \mathbb{P}^1 -bundle over Q' . Since X_1 and X_2 do not go through the point P they are not blown up and we will, by abuse of notation, also denote their pre-images in \tilde{Q} by X_1 and X_2 . The Chow ring of \tilde{Q} is generated by $H = p^*(H_{\mathbb{P}^5})$ and $H' = \pi^*(H_{\mathbb{P}^4})$. Clearly $H^4 = H^3H' = H^2(H')^2 = H(H')^3 = 2$ and $(H')^4 = 0$. Let E be the exceptional divisor of the map p . Then $E = \alpha H + \beta H'$ and from $EH^3 = 0$ and $E(H')^3 = 2$ one deduces $\alpha = 1$ and $\beta = -1$, i.e. $E = H - H'$. The surfaces X_i have class

$$X_i = \alpha_i H^2 + \beta_i H H' + \gamma_i (H')^2.$$

From $X_i H^2 = \deg X_i = 4$ one computes $\alpha_i + \beta_i + \gamma_i = 2$. Since X_i does not meet E we have $X_i E H' = 0$ and from this one deduces $\gamma_i = 0$ and hence

$$X_i = \alpha_i H^2 + (2 - \alpha_i) H H'.$$

But then $X_i^2 = 8$ and this proves the claim. □

We analyze next what happens when X_1 and X_2 lie on a pencil of hyperquadrics.

Proposition 2.3 *Assume that the Veronese surfaces X_1 and X_2 meet in finitely many points and assume that they are contained in a pencil of hyperquadrics $\{\lambda_1 Q_1 + \lambda_2 Q_2 = 0\}$. Then for a general hyperquadric Q in this pencil*

$$X_1 \cap X_2 \cap \text{Sing } Q = \emptyset.$$

Proof. Assume that this is false. Since $X_1 \cap X_2$ is a finite set it follows that there exists some point $P \in X_1 \cap X_2$ which is singular for every hyperquadric Q in this pencil. From what we saw in the proof of Proposition 2.2 this also shows that the general quadric in this pencil has rank at most 4 (and at least 3 since both surfaces X_i are non-degenerate). Projecting from P maps X_1 and X_2 to rational cubic scrolls Y_1 and Y_2 in \mathbb{P}^4 , respectively. These cubic scrolls are contained in a pencil of non-degenerate hyperquadrics $\{\lambda_1 Q'_1 + \lambda_2 Q'_2 = 0\}$ whose general member has rank 3 or 4. For degree reasons this implies $Y_1 = Y_2$. (Incidentally this also shows that X_1 and X_2 are contained in a net of hyperquadrics whose general element has rank 4.)

Let Y be the cone over $Y_1 = Y_2$ with vertex P . We obviously have $X_1, X_2 \subset Y$. We now blow up in P and obtain a diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{p} & Y \subset \mathbb{P}^5 \\ \downarrow \pi & & \\ Y_1 & & \end{array}$$

where \tilde{Y} is a \mathbb{P}^1 -bundle over Y_1 . The Picard group of Y_1 is generated by two elements C_0 and F with $C_0^2 = -1, C_0 F = 1$ and $F^2 = 0$. Let $F_1 = \pi^* C_0$ and $F_2 = \pi^* F_1$. Then the Chow group on \tilde{Y} is generated by $H = p^*(H_{\mathbb{P}^5}), F_1$ and F_2 .

For geometric reasons $H^3 = 3, F_1^2 H = -1, F_1 F_2 H = F_1 H^2 = F_2 H^2 = 1$ and $F_2^2 H = F_1^2 F_2 = F_2^2 F_1 = 0$. Let E be the exceptional locus of p and let \tilde{X}_i denote the strict transforms of the Veronese surfaces. Then π restricted to \tilde{X}_i defines isomorphisms between \tilde{X}_i and Y_1 . Since X_1 and X_2 intersect in only finitely many points and since both are contained in the 3-dimensional cone Y it follows that $X_1 \cap X_2 = \{P\}$, and from this one concludes that $\tilde{X}_1 \cap \tilde{X}_2 = L$ where $L = E \cap F_1$ is a projective line, respectively $\tilde{X}_1 \tilde{X}_2 = aL$ for some $a \geq 1$. Next we want to determine the class of \tilde{X}_i in \tilde{Y} . Since the \tilde{X}_i are sections of the \mathbb{P}^1 -bundle $\pi : \tilde{Y} \rightarrow Y$ we find $\tilde{X}_i = H + \beta_i F_1 + \gamma_i F_2, i = 1, 2$. Restricting this to E and using that H is trivial on E we immediately find that $\beta_i = 1$ and $\gamma_i = 0$, i.e. $\tilde{X}_i = H + F_1$. But then $\tilde{X}_1 \tilde{X}_2 = H^2 + 2H F_1 + F_1^2 \neq aL$ where the latter inequality can be seen e.g. by intersecting with H . This is a contradiction and the proposition is proved. \square

Proposition 2.4 *Let X_1, X_2 be two Veronese surfaces in \mathbb{P}^5 intersecting in a finite number of points. If X_1 and X_2 are contained in a pencil of hyperquadrics, then $X_1.X_2 \leq 10$.*

Proof. Let r be the rank of a general element of this pencil of hyperquadrics. If $r = 6$ or 5 then the assertion follows from Proposition 2.1, or from Proposition 2.2, respectively. On the other hand, since the surfaces X_i are non-degenerate we must have $r \geq 3$.

We shall first treat the case $r = 4$. According to Proposition 2.3 we can then choose a rank 4 hyperquadric Q with $X_1 \cap X_2 \cap \text{Sing } Q = \emptyset$. Blowing up the singular line L of Q we obtain a diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{p} & Q \subset \mathbb{P}^5 \\ \downarrow \pi & & \\ Q' := \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

where π is the structure map of a \mathbb{P}^2 -bundle. We denote the strict transforms of X_i by \tilde{X}_i , $i = 1, 2$. Since $X_1 \cap X_2 \cap \text{Sing } Q = \emptyset$ we have $\tilde{X}_1 \cdot \tilde{X}_2 = X_1 \cdot X_2$. Let $H = p^*(H_{\mathbb{P}^5})$, let L_1, L_2 denote the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and set $F_i = \pi^*L_i$, $i = 1, 2$. Then $H^4 = 2, F_1H^3 = F_2H^3 = F_1F_2H^2 = 1$ and $F_1^2 = F_2^2 = 0$. Let E be the exceptional locus of p . Its class must be of the form $E = H + \alpha_1F_1 + \alpha_2F_2$. From $EF_1H^2 = H^2 = EF_2H^2 = 0$ one deduces $\alpha_1 = \alpha_2 = -1$, i.e. $E = H - F_1 - F_2$.

Now let X be any Veronese surface on Q . We want to determine the possible classes of the strict transform \tilde{X} of X in \tilde{Q} . Let $\tilde{X} = \alpha H^2 + \beta_1F_1H + \beta_2F_2H + \gamma F_1F_2$. From $\tilde{X}H^2 = 4$ we obtain $2\alpha + \beta_1 + \beta_2 + \gamma = 4$. A priori the singular line L can either be disjoint from X , meet it transversally in one point, be a proper secant or a tangent of X . Projection from L shows that only the first and the third of these possibilities can occur.

Assume first that L and X are disjoint. Then $\pi_{|\tilde{X}} : \tilde{X} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a $2 : 1$ map which shows $\alpha = 2$. From $\tilde{X}EF_1 = \tilde{X}EF_2 = 0$ we conclude $\beta_1 = \beta_2 = 0$ and hence $\gamma = 0$, i.e. $\tilde{X} = 2H^2$.

Assume now that L is a proper secant of X . Blowing up Q along L then blows up X in 2 points and the corresponding exceptional curves are mapped to different rulings in $\mathbb{P}^1 \times \mathbb{P}^1$. The map $\pi_{|\tilde{X}} : \tilde{X} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is now birational and hence $\alpha = 1$. From what we have just said it follows that $\tilde{X}EF_1 = \tilde{X}EF_2 = 1$ and hence this implies $\beta_1 = \beta_2 = 1$. But then $\gamma = 0$ and $\tilde{X} = H^2 + HF_1 + HF_2$.

Let $c_1 = 2H^2$ and $c_2 = H^2 + HF_1 + HF_2$. The claim of the proposition now follows for the rank 4 case since $c_1^2 = c_2^2 = c_1c_2 = 8$.

It remains to deal with the case in which the general hyperquadric Q in the pencil has rank 3. By Proposition 2.3 we can again assume that $X_1 \cap X_2 \cap \text{Sing } Q = \emptyset$. Projection from the singular locus of Q gives a diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{p} & Q \subset \mathbb{P}^5 \\ \downarrow \pi & & \\ C \cong \mathbb{P}^1 & & \end{array}$$

where C is a conic section and $\pi : \tilde{Q} \rightarrow C$ is a \mathbb{P}^3 -bundle. We denote $H = p^*(H_{\mathbb{P}^5})$ and $F = \pi^*(pt)$. Then $H^4 = 2, H^3F = 1$ and $F^2 = 0$. Let X be any Veronese surface on Q and denote its strict transform on \tilde{Q} by \tilde{X} . We first note that $X \cap \text{Sing } Q$ is a finite set. Otherwise $X \cap \text{Sing } Q$ would have to be a conic section and projection from $\text{Sing } Q$ would map X onto a plane, not to a conic. Finally let E be the exceptional locus of p . The class of E must be of the form $E = H + \gamma F$ and from $EH^3 = 0$ it follows that $\gamma = -1$, i.e. $E = H - F$. Now put $\tilde{X} = \alpha H^2 + \beta H.F$. From $\tilde{X}H^2 = 4$ one deduces that $2\alpha + \beta = 4$. Since $X \cap \text{Sing } Q$ is finite one must have that $\tilde{X}EH = 0$ and hence $\beta = 0$. This shows that $\tilde{X} = 2H^2$ and the claim of the proposition follows since $X_1.X_2 = \tilde{X}_1.\tilde{X}_2 = 4H^4 = 8$. \square

Remark The above proof shows that if the general element in the pencil of hyperquadrics containing X_1 and X_2 has rank 3 or 4, then $X_1.X_2 = 8$.

3. Catalecticant symmetroids and Veronese surfaces

In this section we prove Coble's claim [Cob, Theorem 26], mentioned in the introduction, that ten points in \mathbb{P}^5 which are the Gale transform of the nodes of a general quartic symmetroid in \mathbb{P}^3 are the simple intersection points of two Veronese surfaces.

A *quartic symmetroid* is the quartic surface in \mathbb{P}^3 defined by the determinant of a symmetric 4×4 matrix with linear entries in \mathbb{P}^3 ; for general choices (of the matrix) the symmetroid has only ordinary double points as singularities (nodes) and their number is 10, by Porteous' formula. These surfaces are sometimes called *Cayley symmetroids*, as Cayley initiated their study in [Cay] (cf. [Jes2], but see [Cos] for a modern account of Cayley's results and much more).

A symmetric matrix whose diagonals are constant is called a *catalecticant* matrix. Surprisingly enough, it turns out that a symmetric 4×4 matrix with linear entries in \mathbb{P}^3 can always be reduced to a catalecticant form (with respect to suitable bases). This fact goes back to Reye [Rey, page 78-79] and Conner [Con, page 39] and is (re)-proved below.

We will make use of the perfect pairing, called *apolarity*, between forms of degree n and homogeneous differential operators of order n induced by the action of $T = k[\partial_0, \dots, \partial_r]$ on $S = k[x_0, \dots, x_r]$ via differentiation:

$$\partial^\alpha(x^\beta) = \alpha! \binom{\beta}{\alpha} x^{\beta-\alpha},$$

if $\beta \geq \alpha$ and 0 otherwise, and where α and β are multi-indices, $\binom{\beta}{\alpha} = \prod \binom{\beta_i}{\alpha_i}$, and k is a field of characteristic zero.

Proposition 3.1 *The Hessian matrix of a web of quadrics in \mathbb{P}^3 is catalecticant (with respect to a suitable basis) if and only*

if the quadrics in the web annihilate the quadrics of a twisted cubic curve. (One says in this situation that the web is “orthic” to the twisted cubic curve.)

Proof. Let $q : W^* \hookrightarrow \text{Sym}_2 V$ be the web of quadrics on $\mathbb{P}^3 = \mathbb{P}(V)$. A twisted cubic $C \subset \check{\mathbb{P}}^3 = \mathbb{P}(V^*)$ is defined by its quadrics $H^0(\check{\mathbb{P}}^3, \mathcal{I}_C(2))$. In suitable coordinates, say $\partial_0, \dots, \partial_3$, these are the minors of the matrix

$$\begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ \partial_1 & \partial_2 & \partial_3 \end{pmatrix}.$$

In terms of the dual coordinates, x_0, \dots, x_3 of $\mathbb{P}(V)$, the web q has the form

$$\begin{aligned} a_0 x_0^2 + a_4 x_1^2 + a_7 x_2^2 + a_9 x_3^2 + 2a_1 x_0 x_1 + 2a_2 x_0 x_2 + 2a_3 x_0 x_3 \\ + 2a_5 x_1 x_2 + 2a_6 x_1 x_3 + 2a_8 x_2 x_3 \end{aligned}$$

where a_0, a_1, \dots, a_9 are linear forms in the variables of W . Direct computation shows that a quadric in the web q is annihilated by the equations $\partial_0 \partial_2 - \partial_1^2, \partial_0 \partial_3 - \partial_1 \partial_2, \partial_1 \partial_3 - \partial_2^2$ if and only if $a_2 = a_4, a_3 = a_5, a_6 = a_7$. It follows that the web q is orthic to the twisted cubic C iff its Hessian matrix has shape

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 & b_4 \\ b_2 & b_3 & b_4 & b_5 \\ b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

where b_0, b_1, \dots, b_6 are linear forms in the variables of W , i.e. it is catalecticant. □

It actually turns out that a 4×4 symmetric matrix with linear entries in \mathbb{P}^3 can be represented in two different ways as a catalecticant matrix. Namely

Proposition 3.2 *There are exactly two twisted cubic curves whose defining quadrics are annihilated by a general web of quadrics in \mathbb{P}^3 .*

Proof. We use the same notation as in the proof of the previous proposition. Namely, let $q : W^* \hookrightarrow \text{Sym}_2 V$ denote a general web of quadrics in $\mathbb{P}^3 = \mathbb{P}(V)$, and let $\check{\mathbb{P}}^3 = \mathbb{P}(V^*)$ denote the dual space. We also choose coordinates as above such

that $S = k[x_0, \dots, x_r]$ and $T = k[\partial_0, \dots, \partial_r]$ are the coordinate rings of $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, respectively. Let now $q' : U \subset W^* \hookrightarrow \text{Sym}_2(V)$ be a general subnet. The variety $H(q')$ of twisted cubics in $\mathbb{P}(V^*)$ whose defining quadratic equations are annihilated by the net q' is the geometric realization of a prime Fano threefold X_q of genus 12 (see [Muk1] and [Sch]). Via the apolarity pairing, $X_q \subset \text{Gr}(\mathbb{P}^2, \mathbb{P}(U_q))$, where the annihilator $U_q = (q^\perp)_2 \subset \text{Sym}_2 V^*$ is a 7-dimensional vector space. As a subvariety of the Grassmannian the Fano threefold X_q is the (codimension 9) common zero-locus of three sections of $\wedge^2 \mathcal{E}$, where \mathcal{E} is the dual of the tautological subbundle on $\text{Gr}(\mathbb{P}^2, \mathbb{P}(U_q))$, cf. [Muk1] or see [Sch, Theorem 5.1] for a complete proof.

The choice of a (general) subnet q' is equivalent to the choice of a (general) global section of \mathcal{E} . But \mathcal{E} is a globally generated rank 3 vector bundle whose restriction $E = \mathcal{E}|_{X_q}$ has third Chern number 2, as an easy direct computation shows. By Kleiman's transversality theorem the (general) section of E corresponding to q' must vanish exactly at two (simple) points of X_q . These in turn correspond to two twisted cubic curves each of whose defining quadrics are annihilated not only by the net q' , but by the whole web q (the zero locus of a section in \mathcal{E} is the special Schubert cycle of subspaces lying in the hyperplane dual to the section). This concludes the proof. \square

Let $C \subset \mathbb{P}^6$ be a rational normal sextic curve, and let $S = \text{Sec}(C) \subset \mathbb{P}^6$ be its secant variety. S has degree 10, since this is the number of nodes of a general projection of C to a plane. The homogeneous ideal of C is generated by the 2×2 -minors of either a 3×5 or a 4×4 catalecticant matrix with linear entries, induced by splittings of $\mathcal{O}_{\mathbb{P}^1}(6)$ as a tensor product of two line bundles of strictly positive degree. Furthermore, it is known that the homogeneous ideal of $S = \text{Sec}(C)$ is generated by the 3×3 minors of either of the above two catalecticant matrices (see [GP] or [EKS]).

For $\Pi = \mathbb{P}^3 \subset \mathbb{P}^6$ a general 3-dimensional linear subspace, the linear section $\Gamma = \text{Sec}(C) \cap \Pi$ consists of 10 simple points in \mathbb{P}^3 defined by the 3×3 minors of a 4×4 symmetric (even catalecticant) matrix with linear entries in the variables of Π . Conversely, by Proposition 3.2 above, the set $\Gamma \subset \mathbb{P}^3$ of 10 nodes of a general quartic symmetroid in \mathbb{P}^3 arises always as a linear section of the secant variety of the rational normal curve in \mathbb{P}^6 (in two different ways). Moreover, it follows that Γ can also be defined by the 3×3 -minors of each of two different 3×5 catalecticant matrices with linear entries in \mathbb{P}^3 . Since the 2×2 minors of these catalecticant matrices generate an irrelevant ideal, we may apply [EiPo, Theorem 6.1] (see also [EiPo, Example 6.3] for more details) to obtain the following

Proposition 3.3 (Coble) *The Gale transform of the 10 nodes of a general quartic symmetroid in \mathbb{P}^3 are the points of intersection of two Veronese surfaces in \mathbb{P}^5 .*

Remark 1) A more careful analysis of the preceding argument shows that the needed generality assumptions on the quartic symmetroid are satisfied if the quartic symmetroid is defined by a regular web of quadrics in \mathbb{P}^3 , see [Cos, Definition 2.1.2].

2) Coble asserts in [Cob] that the converse to Proposition 3.3 should also be true, presumably under suitable generality assumptions. This also relates to the question mentioned in [EiPo] of describing when a collection of 10 points in \mathbb{P}^3 are determinantal.

4. Further results

One may now ask which intersection numbers can actually occur and in which geometric situation this can happen. We are far from having a complete answer to this question, but want to state a number of results in this direction.

We start by considering Veronese surfaces which intersect in 10 points. It is easy to find examples of surfaces X_1 and X_2 intersecting transversally in 10 points. For this, one can start with an arbitrary surface $X_1 \subset \text{Gr}(\mathbb{P}^1, \mathbb{P}^3) \subset \mathbb{P}^5$. For a general automorphism φ of \mathbb{P}^3 the surface $X_2 = \varphi(X_1)$ intersects X_1 transversally by Kleiman's transversality theorem and since both X_1 and X_2 have the same cohomology class we have $X_1 \cdot X_2 = 10$. Actually, we have the following

Proposition 4.1 *Let X_1, X_2 be two Veronese surfaces in \mathbb{P}^5 intersecting in 10 points. Then $X_1 \cup X_2$ is contained in a hyperquadric Q and one of the following cases occurs:*

- (i) Q has rank 6 and X_1 and X_2 lie in the same cohomology class,
- (ii) The rank of Q is 4 and $X_1 \cap X_2 \cap \text{Sing } Q \neq \emptyset$,

Proof. The existence of Q follows from Proposition 1.3. If Q has rank 6 then X_1 and X_2 must have the same class by the proof of Proposition 2.1. The case rank $Q = 5$ is excluded by Proposition 2.2 and the case of rank $Q \leq 4$ and $X_1 \cap X_2 \cap \text{Sing } Q = \emptyset$ is excluded by the remark at the end of Section 2. We are now left to exclude the case where rank $Q = 3$ and $X_1 \cap X_2 \cap \text{Sing } Q \neq \emptyset$. We will make use of the diagram and the computations at the end of the proof of Proposition 2.4.

For a Veronese surface $X \subset Q$ with rank $Q = 3$ the class of \tilde{X} in \tilde{Q} equals $2H^2$. The fibres of the map $\pi|_{\tilde{X}} : \tilde{X} \rightarrow C \cong \mathbb{P}^1$ are conics and hence this linear system is a subsystem of $|2l|$ on $X \cong \mathbb{P}^2$, i.e. contained in some system of the form $|2l - \sum \alpha_i P_i|$. We have $(2l - \sum \alpha_i E_i)^2 = 4 - \sum \alpha_i^2 = 0$. This implies either $\alpha_1 = \dots = \alpha_4 = 1$ or $\alpha_1 = 2$. But by the argument in the proof of Proposition 2.4 we already know that X intersects the vertex of Q in a finite non-empty set of points. Since the fibres of the map $\pi|_{\tilde{X}}$ are conics the first case can only occur if we have the linear system of conics through 4 points in general position. In this case \tilde{X} is mapped to a \mathbb{P}^1 and the general fibre is an irreducible conic, which contradicts what we have. This implies that the linear system is given by $|2l - 2P|$, in particular X meets the vertex of Q in exactly one point P and that this intersection is not-transversal.

Returning to the case we have to exclude, we can assume that X_1 is given by the 2×2 -minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_3 & x_4 & x_5 \end{pmatrix}$$

and so a typical rank 3 hyperquadric through X_1 is $Q = \{x_0x_3 - x_1^2 = 0\}$. The vertex V of Q is the plane $\{x_0 = x_1 = x_3 = 0\}$. The intersection of X_1 and V is then defined by the ideal (x_2x_4, x_2^2, x_4^2) , i.e. the first infinitesimal neighborhood of P . The same holds for the second Veronese surface $X_2 \subset Q$. Since both surfaces \tilde{X}_1 and \tilde{X}_2 have class $2H^2$ on \tilde{Q} , it follows that \tilde{X}_1 and \tilde{X}_2 must meet in 6 other points (different of P). If these points are all in different fibers of π , then X_1 and X_2 meet in a (non-reduced) scheme of length $9(= 6 + 3)$ and thus we got a contradiction. Otherwise, since all the fibers of $\pi|_{\tilde{X}_i}$ are conics (whose images in \mathbb{P}^5 already go through the fixed point P), the 2-planes spanned by the fibres of $\pi|_{\tilde{X}_1}$ and $\pi|_{\tilde{X}_2}$ over some point in C must coincide. But then, by Bezout, the two conic fibres must intersect in 4 points, from which we conclude $X_1.X_2 \geq 11$, again impossible. This concludes the proof of the proposition. \square

Remark As corollary of the proof of the previous proposition, we find out that two Veronese surfaces in \mathbb{P}^5 may intersect in a non-reduced scheme of length 9. This is indeed the case, for two (general) Veronese surfaces X_1 and X_2 lying on a rank 3 hyperquadric Q such that $X_1 \cap X_2 \cap \text{Sing } Q$ consists of a single point P . As above, both surfaces cut out the first infinitesimal neighborhood of P on the vertex of the hyperquadric Q and meet further in 6 simple points. In particular their intersection is non-reduced.

However, we show that

Proposition 4.2 *Two Veronese surfaces in \mathbb{P}^5 cannot intersect transversally in 9 points.*

Proof. Assume that there exist Veronese surfaces X_1 and X_2 in \mathbb{P}^5 such that $W = X_1 \cap X_2$ is a reduced set of length 9. We consider again W as a subscheme of $X_1 \cong \mathbb{P}^2$ and we recall from Lemma 1.1 that $h^1(\mathcal{I}_W(4)) = 0$. Equivalently, this means that the linear system $\delta := |4l - W|$ on $X_1 \cong \mathbb{P}^2$ has projective dimension 5. It will then be enough to show that the linear system $|4l - W|$ of plane quartics through W contains a smooth curve C , since in this case the restriction of $|4l - W|$ defines, via taking the residual intersection, a $g_7^{4+\varepsilon}$ on C with $\varepsilon \geq 0$, thus contradicting Clifford's theorem.

In order to show the existence of such a C we consider the surface S given by blowing up the set W on X_1 . It will be enough to check that the linear system $|4l - W|$ is base point free and defines a morphism $S \rightarrow S' \subset \mathbb{P}^5$ whose image S' has no

worse than isolated singularities. We want to do this using Reider's theorem (see [BS, Theorem 2.1] in characteristic 0, and [SB], [Nak], [Ter, Theorem 2.4] in positive characteristic) and for this purpose we write $4l - W = L + K_\delta$ where $K_\delta = -3l + W$ is the canonical divisor on S while $L = 7l - 2W$. In order to apply a Reider type theorem we need to check that $L^2 \geq 9$ and that L is nef and big. The first is clear since $L^2 = 49 - 36 = 13$. We do not know that L is nef and big, but in the proof of Reider's theorem (as in the proof of its positive characteristic counterparts, cf [Ter]) this assumption is only used to conclude that $h^1(K_\delta + L) = 0$, which we already know since $h^1(\mathcal{I}_W(4)) = 0$ by Lemma 1.1.

We may now argue as follows. If $|4l - W| = |K_\delta + L|$ is not base-point free, respectively very ample, then there exists a curve a D such that $L - 2D$ is \mathbb{Q} -effective and such that

$$D^2 \geq L.D - k - 1$$

where $k = 0$, respectively 1. We write $D = al - \sum_{i=1}^9 b_i E_i$. Since $L - 2D$ must be \mathbb{Q} -effective we see immediately that $a \leq 3$.

For $a = 3$ we obtain

$$g - \sum_{i=1}^9 b_i^2 \geq 21 - 2 \sum_{i=1}^9 b_i - k - 1$$

respectively

$$\sum_{i=1}^9 b_i(b_i - 1) \leq -2 + k$$

which gives a contradiction. For $a = 1, 2$ the same calculation gives

$$\sum_{i=1}^9 (b_i - 1)^2 \leq 4 + k \quad (a = 1),$$

respectively

$$\sum_{i=1}^9 (b_i - 1)^2 \leq k.$$

On the other hand, since the quadrics through X_2 cut out X_2 , then at most 4 points of $W \subset \mathbb{P}^2$ can be collinear and at most 8 points of W can lie on the same conic. This shows that $|4l - W|$ is base point free on S and that S' has at most isolated singularities, which is our claim, and this concludes the proof of the proposition. \square

Remark One may construct pairs of Veronese surfaces on suitable smooth or nodal cubic hypersurfaces in \mathbb{P}^5 which meet in 1, 2, 3, 5 or 6 simple points. It is also possible to check in *Macaulay* [Mac] that if $X \subset \mathbb{P}^5$ is a Veronese surface and φ is a general linear automorphism of \mathbb{P}^5 fixing $m \in \{1, 2, 3, 5\}$ (general) points on X , then X and $\varphi(X)$ meet exactly at those m points.

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