Homework 8

Section 2.3

See pg302 for 2c, 2f, 6, 7.

2(d). Let $R$ be a relation on $X \times Y$. The complement, $R^c$, of $R$ is also a relation on $X \times Y$. Suppose that $(a, b) \in R^c$. If $(b, a)$ is not in $R^c$, then $(b, a) \in R$. Since $R$ is symmetric, we have $(a, b) \in R$. Thus $(a, b)$ is not in $R^c$ which is a contradiction to our assumption. Therefore $(b, a) \in R^c$ and $R^c$ is symmetric.

2(e). Let $R$ be a relation on $X \times Y$. The reverse $R^r \subset Y \times X$ of $R$ is defined to be the set

$$\{(a, b) | (b, a) \in R\}.$$ 

If $(a, b) \in R^c$ and $(b, c) \in R^r$ then, by definition, $(b, a) \in R$ and $(c, b) \in R$. By assumption, $R$ is transitive and $(c, b), (b, a)$ are both in $R$, so we have $(c, a) \in R$. Thus $(a, c) \in R^r$, which concludes the proof.

4. Take a point in each country and join two points by an edge if the two countries represented by those points have a common border. The result is a planar graph. See the following website for further information about the Four Color Theorem:

http://www.math.gatech.edu/~thomas/FC/fourcolor.html

Section 4.1 1,2,3 see pp. 307-308

– Let $X = 1,2,3,4,5$. For each part, define a relation $R$ on $X$ satisfying

1. $R$ is reflexive and symmetric, but not transitive

One example that works here is $xRy$ iff $|x - y| \leq 1$. Reflexivity and symmetry are easy to check, and so is the failure of transitivity: $1R2$ and $2R3$ are true, but $1R3$ is false.
2. R is symmetric and transitive, but not reflexive

Suppose xRy for any x,y in X. Then, by symmetry, yRx, so by transitivity xRx. If this were true of every x, then R would be reflexive, so there must be at least one z in X so that zRy is never true for any y in X. One example is the empty relation: xRy is never true. Another is the relation given by xRy iff neither x nor y equals 5.

3. R is reflexive, symmetric, transitive, and weakly anti-symmetric

Suppose xRy. Then symmetry gives us yRx, and weak antisymmetry gives us x=y, so xRy implies x=y. Reflexivity tells us that x=y implies xRy. Thus, xRy iff x=y, so R must be =, and we see that this works: = is reflexive, symmetric, transitive, and weakly antisymmetric.

– Let $M$ be the relation on the real numbers $\mathbb{R}$ defined as follows: for all $x, y \in \mathbb{R}$, $xMy$ if and only if $x - y$ is an integer.

$M$ is an equivalence relation: $x - x = 0 \in \mathbb{Z}$ so $M$ is reflexive, if $x - y \in \mathbb{Z}$ then $y - x \in \mathbb{Z}$ so $M$ is symmetric, and if $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$, then $x - z = (x - y) + (y - z) \in \mathbb{Z}$ so $M$ is transitive.

The equivalence classes of $M$ correspond to all possible fractionary parts of real numbers. Thus there is an equivalence class for every number $t \in [0, 1)$. Equivalently the collection of equivalence classes of $M$ form a circle (the real line wraps infinitely many times around it).