# Homework 6

Compute $\phi(n)$ for the following values of $n = 10!, 20!$ and $100!$; explain the method you have used.

Each prime $p < n$ will occur in the product $n!$ once for each multiple of any power of p less than n. For example, 2 occurs 8 times in 10!, 5 for the five multiples of 2: (2,4,6,8,10), plus 2 more for those multiples of 2 which are also multiples of $2^2$: (4,8), plus 1 more for those multiples of $2^2$ which are also multiples of $2^3$: (8). Thus, we count the occurrences of a prime p by dividing by its various powers until we get 0.

For 10, we get

2: $5 + 2 + 1 = 8$

3: $3 + 1$

5: 2

7: 1

Thus $10! = 2^8 3^4 5^2 7^1$, so $\phi(10!) = 2^7(2-1)3^3(3-1)5(5-1)(7-1) = 967680$

For 20, we get

2: $10 + 5 + 2 + 1 = 18$

3: $6 + 2 = 8$

5: 4

7: 2

11, 13, 17, 19: 1

so $\phi(20!) = (2^{17})(3^7(2))(5^3(4))(7(6))(10)(12)(16)(18)$

A similar method will work for 100! (note that you need check higher powers of primes only up through 10).

Section 2.1

1, 4, 6 see pg300.

3. We first show that $X \backslash Y$ is a subset of $X \cap Y^c$. For any $p \in X \backslash Y$, $p \in X$ and $p \notin Y$, so $p \in X$ and $p \in Y^c$. This means $p \in X \cap Y^c$. For the converse, that is $X \cap Y^c$ is a subset of $X \backslash Y$, we see that for any $p \in X \cap Y^c$, $p$ is in $X$ but $p$ is not in $Y$, so $p \in X \backslash Y$.

Section 2.2

1, 2, 5 see pg301.

9. Let $x$ be the number of students who take both mathematics, physics and chemistry classes. The total number of students who take at least one of the above classes is 50-12=38. By using the formula

$$|M \cup P \cup C| = |M| + |P| + |C| - |M \cap P| - |M \cap C| - |P \cap C| + |M \cap P \cap C|,$$

we get

$$38 = 23 + 14 + 17 - 5 - 3 - 7 + x.$$

A simple calculation shows that $x = -1$ which is a contradiction since $x$ must be positive.

10. (a) If $A = B$, then surely they have the same characteristic function. Conversely, if $\chi_A = \chi_B$, then the sets of these two functions take value 1 (that is $A$ and $B$) have to coincide.

(b) Let $f : X \longrightarrow \{0, 1\}$ be a function. Let

$$Y = \{x \in X | f(x) = 1\},$$

so $Y$ is a subset of $X$. Consider the characteristic function $\chi_Y$ of $Y$. Then for any $p \in Y$,

$\chi_Y(p) = f(p) = 1$, and for any $p \notin Y$, $\chi_Y(p) = 0 = f(p)$. $\chi_Y$ and $f$ take the same values for any point in $X$, so they must be the same function.

Section 2.3

1 see pg301