Homework 3

Section 1.4

3, 4, 9: see pg 298.

5. Assume that there is an integer of the form $8n+7$ with $8n+7 = a^2+b^2+c^2$ for some integers a, b, c. Considering the congruence classes of these numbers in $\mathbb{Z}_8$ yields $[8n+7]_8 = [7]_8 = [a^2]_8 + [b^2]_8 + [c^2]_8$. Every class in $\mathbb{Z}_8$ has as square either $[0]_8, [1]_8$ or $[4]_8$, while it is easy to see that the sum of any three elements (not necessarily distinct) from the set $\{[0]_8, [1]_8, [4]_8\}$ will never equal to $[7]_8$. This contradicts our hypothesis that $[a^2]_8 + [b^2]_8 + [c^2]_8 = [7]_8$, so no number of the form $8n+7$ is the sum of three squares.

Section 1.5:

For Ex 1, 2 see pg 299.

3. The conditions in the statement of the exercise are equivalent to the following system of congruences:
$$\begin{cases} x \equiv 8 \mod 11 & (1) \\ x \equiv 4 \mod 10 & (2) \\ x \equiv 0 \mod 27 & (3) \end{cases}$$

There are several ways to solve this system. For instance we may first solve (1) and (2). Since $(11, 10) = 1$, and actually

$$(11)(1) + (10)(-1) = 1,$$

so the formula learned in class yields

$$x \equiv (8)(10)(-1) + (4)(11)(1) \mod 110$$

$$\equiv -80 + 44 \mod 110$$

$$\equiv -36 \mod 110$$

$$\equiv 74 \mod 110.$$

We are now left to solve the system $\begin{cases} x \equiv 74 \mod 110 \\ x \equiv 0 \mod 27 \end{cases}$

It is easily checked that $(110, 27) = 1$. Moreover the euclidean algorithm yields an expression for 1 as a linear combination of 110 and 27:

$$\left(\begin{array}{cc|c} 1 & 0 & 110 \\ 0 & 1 & 27 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 1 & 27 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -4 & 2 \\ -13 & 53 & 1 \end{array}\right)$$

Thus

$$(110)(-13) + (27)(53) = 1.$$

Again, we deduce this time that

$$x \equiv (74)(27)(53) + (0)(110)(-13) \mod (110)(27)$$

$$\equiv 105894 \mod 2970$$

$$\equiv 1944 \mod 2970$$

So the answer is 1944.

Non-Book Problems

1. $60x + 18y = 97$ If this had an integral solution, the left side would be even and the right side odd. Thus, no solution exists.

2. $21x + 14y = 147$ Dividing both sides by 7 yields $3x + 2y = 21$. For every odd x, $21 - 3x$ will be even, so there will be an integer $y = (21 - 3x)/2$; For every even x,

the left hand side of the original equation will be even and the right hand side odd. Thus, our solutions are of the form $x = 2k + 1$ and $y = 10 - k$, for every integer k.

Change for a dollar.

With dimes and quarters, the problem is simple enough: find positive integral solutions to $25q + 10d = 100$. Dividing by 5, we get $5q + 2d = 20$. It is easy to see that, similarly to the previous problem, $20 - 5q$ will be divisible by 2 only when q is even, so there will be an integral d if and only if $q = 2k$ for some k. Then our solutions are $q = 2k$ and $d = 10 - 5k$. The only values of k making both of these numbers nonnegative are 0, 1, and 2, so the only possible was of making change are (0 quarters, 10 dimes), (2 quarters, 5 dimes) and (4 quarters, 0 dimes).

When we add nickels, the problem is trickier. A simple (although not very elegant) solution is this. Again dividing by 5, we have $5q + 2d + n = 20$ We know that q can take values from 0 to 4, and our problem is trivial for each of these values, so we can just add them up: For $q = 0$, $2d = 20 - n$, so we get $n = 2k$ and $d = 10 - k$ as k ranges from 0 to 10, yielding 11 solutions. For $q = 1$, $2d = 15 - n$, so we get $n = 2k + 1$ and $d = 7 - k$ as k ranges from 0 to 7, yielding 8 solutions. For $q = 2$, $2d = 10 - n$, so we get $n = 2k$ and $d = 5 - k$ as k ranges from 0 to 5, yielding 6 solutions. For $q = 3$, $2d = 5 - n$, so we get $n = 2k + 1$ and $d = 2 - k$ as k ranges from 0 to 2, yielding 3 solutions. And finally, for $q = 4$, $2d = 0 - n$, so $n = d = 0$ and we get 1 solution. Adding these up gives us 29 distinct ways of making change.