Homework 2

Section 1.3

1, 3, 4: See pg 297

6. If $n$ is not a prime, then there are two positive integers $a, b$, with $a > 1, b > 1$, such that $n = ab$. Thus $2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + \cdots + 1)$. Since $a$ and $b$ are both greater than 1, we have that $2^a - 1$ and $(2^a)^{b-1} + \cdots + 1$ are also greater than 1. Thus $2^n - 1$ is not a prime, a contradiction. Therefore, $n$ must be prime.

7. Let $n = 2^a b$ where $b$ is an odd number and $a$ is a nonnegative integer. Assume $b > 1$. Then $2^n + 1 = 2^{2^a b} + 1 = (2^{2^a})^b + 1 = (2^{2^a} + 1)((2^{2^a})^{b-1} - (2^{2^a})^{b-2} + \cdots + 1)$. Since $2^{2^a} + 1 > 1$ and $b > 1$, we get $(2^{2^a})^{b-1} - (2^{2^a})^{b-2} + \cdots + 1 > 1$. Thus $2^n + 1$ is not a prime, and so so $n = 2^a$.

8. Assume there are only finitely many primes of the form $4k + 3$. Denote them $p_1 = 3, p_2 = 7, \ldots, p_n$, where $p_n$ is the largest prime of form of $4k + 3$. Let now $N := 4(p_2 \times \cdots \times p_n) + 3$. It is not difficult to see that $N$ is not divisible by any of the primes $p_1, \ldots, p_n$. On the other hand any odd integer is either of the form $4k + 1$ or $4k + 3$, and since $N$ is not divisible by any prime $p_1, p_2, \ldots, p_n$, we may write $N = (4a_1 + 1)(4a_2 + 1) \cdots (4a_m + 1)$ for suitable positive integers $a_1, \ldots, a_m$. Expanding the right hand side we may write $(4a_1 + 1)(4a_2 + 1) \cdots (4a_m + 1) = 4b + 1$ for some integer $b$. Thus

$$4(p_2 \times \cdots p_n) + 3 = 4b + 1,$$
an hence

\[ 2 = 4(b - p_2 \times \cdots \times p_n). \]

This implies that

\[ 1 = 2(b - p_2 \times \cdots \times p_n), \]

which is a contradiction since the right hand side is an even number and the left hand side is an odd number. Therefore there exist infinitely many prime numbers of the form \( 4k + 3 \).

Section 1.4

1, 2. See pgs 297-298.