Homework 11

4.4

7. - see p. 311

17. First, we must show $a = a \wedge b \Leftrightarrow b = a \vee b$.

($\Rightarrow$)

$$a = a \wedge b$$

$$a \vee b = (a \wedge b) \vee b$$

$$a \vee b = (a \wedge b) \vee (1 \wedge b)$$

$$a \vee b = (a \vee 1) \wedge b$$

$$a \vee b = 1 \wedge b$$

$$a \vee b = b$$

($\Leftarrow$)

$$b = a \vee b$$

$$a \wedge b = a \wedge (a \vee b)$$

$$a \wedge b = (a \wedge a) \vee (a \wedge b)$$

$$a \wedge b = a \vee (a \wedge b)$$

$$a \wedge b = (a \wedge 1) \vee (a \wedge b)$$

$$a \wedge b = a \wedge (1 \vee b)$$

$$a \wedge b = a \wedge 1$$

$$a \wedge b = a$$

Now we must show that $\leq$ is a partial order on B and that $0 \leq a \leq 1 \qquad \forall a \in B$.

Reflexivity and transitivity are almost immediate:

$$a \wedge a = a \qquad \forall a$$

and $a = a \wedge b$ together with $b = b \wedge c$ yield

$$a \wedge c = a \wedge b \wedge c = a \wedge b = a$$

Weak Antisymmetry is also very easy. Suppose we have $a = a \wedge b$ and $b = b \wedge a$. Then, since $a \wedge b = b \wedge a$, we must have $a = b$. And then $0 \wedge a = (\neg a \wedge a) \wedge a = \neg a \wedge a = 0$ and $a \wedge 1 = a$, so $0 \leq a \leq 1 \quad \forall a$.

On the Boolean algebra of sets, our partial order corresponds to inclusion.

To define the Boolean algebra structure in terms of the partial order structure, basically you want to set $a \vee b$ to be the smallest element c satisfying $a \leq c$ and $b \leq c$, $a \wedge b$ to be the largest element d satisfying $d \leq a$ and $d \leq b$, and $\neg a$ to be the unique set f satisfying both $f \wedge a = 0$ and $f \vee a = 1$. Showing that existence and uniqueness of these definitions, and that they give you back the original boolean structure in full, is a trivial but exhausting exercise which I will omit.

5.1

1. $\Rightarrow$: Suppose that for any elements a and b in G, $ab = ba$. Then $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$.

$\Leftarrow$: Suppose that for any elements a and b in G, $(ab)^{-1} = a^{-1}b^{-1}$. Then $(ba)(ab)^{-1} = (ba)(a^{-1}b^{-1}) = e$. Multiple (ab) to the equation $(ba)(ab)^{-1} = e$ both side from the right, we get $ba = ab$.

3,4,5. - see p. 311