SKETCH OF SOLUTIONS (MIDTERM EXAM)

1.- For each of the congruences below, find all solutions (if any).

(a) $927x \equiv 4 \mod 102$

Notice that $927 \equiv 9 \mod 102$ *and* $(9, 102) = 3$. *Since* 4 *is not a multiple of* $3 \mod 102$ *there are no solutions.*

(b) $928x \equiv 4 \mod 102$

Now we must solve $10x \equiv 4 \mod 102$. *Since* $(10, 102) = 2$ *and* $x \equiv 31 \mod 51$ *the solutions are* $x \equiv 31 \mod 102$ *and* $x \equiv 31 + 51 \equiv 82 \mod 102$

2.- A Japanese tourist returning home from a trip to Europe and U.S. exchanges his Euro and Dollar bills for yens. If he receives 15,286 yen, and received 112 yen for each Euro and 122 for each U.S. Dollar, and he had more Dollars than Euros, how many of each type of currency did he exchange?

Let e *denote the number of euros and* d *the number of dollars, then we must solve the following equation:*

$$112e + 122d = 15286$$

with the restrictions $e, d \geq 0$ *and* $d > e$

Since $122/2 = 61$ *which is prime we have* $(122, 112) = 2$ *Using the euclidean algorithm we find*

$$112(12) - 122(11) = 2$$

Therefore

$$112(12)(7643) - 122(11)(7643) = 2 * 7643 = 15286$$

So all solutions are of the form

$$d = -84073 + 56t, \ \ e = 91716 - 61t$$

Now, e *is positive if* $t > 1501$ *and* d *is positive if* $t < 1504$ *therefore the only possible solutions are* $d = 39$, $e = 94$ *and* $d = 95$, $e = 33$. *But we know* $d > e$ *therefore the solution is* $d = 95$, $e = 33$

3.- What is the smallest natural number n such that

$$\begin{aligned} n &\equiv 1 \mod 3 \\ n &\equiv 3 \mod 8 \\ n &\equiv 2 \mod 5 \end{aligned}$$

Using the Chinese remainder theorem we find the solution:

$$x = (1)(8 * 5)(1) + (3)(3 * 5)(7) + (2)(8 * 3)(4) \equiv 67 \mod 120$$

4.-

(a) What is the last digit in the decimal representation of 7^{19522}?
We are asked to find the smallest natural number which represents the class of $7^{19522} \mod 10$. *Notice that*

$$7^2 \equiv 9 \mod 10$$

$$7^3 \equiv 7 * 7^2 \equiv 7 * 9 \equiv 3 \mod 10$$

and finally

$$7^4 \equiv 7 * 7^3 \equiv 7 * 3 \equiv 1 \mod 10$$

Therefore the remainder we are looking for only depends on the equivalence class of $19522 \mod 4$. *But* 19522 *can only be divided once by* 2, *therefore* $19522 \equiv 2 \mod 4$. *Therefore* $7^{19522} \equiv 7^2 \equiv 9 \mod 10$

(b) Find all the solutions to the congruence

$$x^2 + x \equiv 0 \mod 437$$

$x^2 + x \equiv x(x+1) \mod 437$. *Also, using the fact that* $437 = 23 * 19$ *we find first all the solutions modulo* 23 *and modulo* 19 *which are* $0, 18, 22$ *Now we solve the systems of congruences*

$$\begin{aligned} x &\equiv 18 \mod 19 \\ x &\equiv 22 \mod 23 \end{aligned}$$

$$\begin{aligned} x &\equiv 0 \mod 19 \\ x &\equiv 22 \mod 23 \end{aligned}$$

$$\begin{aligned} x &\equiv 18 \mod 19 \\ x &\equiv 0 \mod 23 \end{aligned}$$

$$\begin{aligned} x &\equiv 0 \mod 19 \\ x &\equiv 0 \mod 23 \end{aligned}$$

and we get all possible solutions, namely $x \equiv 436, 114, 322, 0 \mod 437$

5.- Find at least one solution to the following congruence:

$$x^2 - 3x - 7 \equiv 0 \mod 27$$

We start by looking for solutions $\mod 3$. *Let* $f(x) = x^2 - 3x - 7$ *Then* $f(x) \equiv (x+1)(x-1) \mod 3$ *therefore* $f(1) \equiv 0 \mod 3$ *Using the fact that* $f'(1) \neq 0 \mod 3$, *by Hensel's lemma* $f(1+0) \equiv 0 \mod 3^2$. *Again,* $f(1) \neq 0 \mod 3^2$ *and* $f'(1) \neq 0 \mod 3$ *therefore* $f(1 + 2 * 9) = 0 \mod 3^3$ *i.e.* 19 *is a solution of the congruence. (the other possible solution is* 11*)*

6.- (a) Determine if th following ISBN number is valid:

$$0 - 404 - 50874 - 9$$

Not valid:

$$(1)(0)+(2)(4)+(3)(0)+(4)(4)+(5)(5)+(6)(0)+(7)(8)+(8)(7)+(9)(4)+(10)(9) \equiv 287 \equiv 1 \mod 11$$

(b) While copying the ISBN for a book, a clerk accidentally transposed two digits. If the clerk copied the ISBN as 0-07-289095-0 and did not make any other mistakes, what is the correct ISBN for the book?
Let $x_1 \dots x_{10}$ *be the digits of the correct ISBN, and let* $y_1 \dots y_{10}$ *be the digits of the given ISBN. Since* $\sum_{i=1}^{10} iy_i \equiv 9 \mod 11$ *and* $\sum_{i=1}^{10} ix_i \equiv 0 \mod 11$ *we must have* $\sum_{i=1}^{10} i(y_i - x_i) \equiv 9 \mod 11$ *but we know that* $x_i = y_i$ *for all* i *except for two values* j, k *which are transposed.*

Therefore all terms in the last sum are zero with the exceptions of the terms corresponding to j *and* k *i.e.*

$$j(y_j - x_j) + k(y_k - x_k) \equiv 9 \mod 11$$

We also know that $y_j = x_k$ *and* $y_k = x_j$ *therefore we get the equation*

$$j(y_j - y_k) + k(y_k - y_j) \equiv (y_j - y_k)(j - k) \equiv 9 \mod 11$$

By trial, we find that $j = 8$ *and* $k = 7$ *work:*

$$(y_8 - y_7)(8 - 7) \equiv (9)(1) \equiv 9 \mod 11$$

Therefore the correct ISBN is $0 - 07 - 289905 - 0$