# SKETCH OF SOLUTIONS (HOMEWORK IX)

2.- a) $\tau(36) = \tau(2^2 3^2) = (2+1)(2+1) = 9$, b) $\tau(99) = \tau(3^2 \cdot 11) = (2+1)(1+1) = 6$, c) $\tau(144) = \tau(2^4 \cdot 3^2) = (4+1)(2+1) = 15$

21.- $\sigma_k(p) = 1 + p^k$

22.- $\sigma_k(p^a) = \frac{p^{k(a+1)}-1}{p-1}$

23.- Let $a, b$ be such that $(a, b) = 1$. Then $\sum_{d|ab} d^k = \sum_{d|a,d|b} (d_1 d_2)^k = \sum_{d|a} d_1^k \sum_{d_2^k} = \sigma_k(a)\sigma_k(b)$

34.- Both sides of the equality are multiplicative functions, therefore we only need to verify that they coincide for $n = p^k$ with $p$ prime and $k$ a positive integer. But in this case:

$$\left(\sigma_{d|p^k} \tau(d)\right)^2 = \left(\sum_{j=0}^{k} \tau(p^j)\right)^2 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

(using the formula for the sum of the first $k + 1$ positive integers). Also,

$$\sum_{d|p^k} \tau(d)^3 = \sum_{j=0}^{k} \tau(p^j)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

(using the formula for the sum of the cubes of the first $k+1$ positive integers)

37.- Let $M$ be the matrix with entries $(i, j)$. Let $D$ be the matrix with entries $\Phi(1), \Phi(2), \ldots, \Phi(n)$ on the diagonal and zeros elsewhere. Let $A$ be the matrix defined by the rule: If $i$ divides $j$ then the $(i, j)$-th entry is 1 and it is zero otherwise. Notice that $A$ has only zeros below the main diagonal, therefore $\det(A) = \det(A^t) = 1$. Also notice that $D = ADA^t$ (to prove this, use the identity $\sum_{k|(i,j)} \Phi(k) = (i, j)$ and the fact that if $k \mid i$ and $k \mid j$ then $k \mid (i, j)$). Since $\det(D) = \det(ADA^t) = 1 \cdot \det(D) \cdot 1 = \Phi(1)\Phi(2) \cdots \Phi(n)$ we are done.

**Section 7.4**

1.- a) 0, b) 1

15.- Using Moebius inversion formula with the identity $n = \sum_{d|n} \Phi(d)$ we get that $\Phi(n) = n \sum_{d|n} \mu(d)/d$

17.- Since $f$ and $\mu$ are multiplicative, so is $f\mu$ and also $\sum_{d|n} \mu(d)f(d)$. Therefore it suffices to prove the theorem for prime powers. But

$$\sum_{d|p^a} \mu(d)f(d) = \mu(p^a)f(p^a) + \ldots \mu(p)f(p) + \mu(1)f(1) = -f(p) + 1$$

(Since $\mu(p^j) = 0$ for $j > 1$)

18.- Using $f(n) = n$ in exercise 17 we get $\sum_{d|n} d\mu(d) = \Pi_{i=1}^{k}(1 - p_i)$

23.- Using $f(n) = \mu(n)$ in exercise 17 we get $\sum_{d|n} \mu(d)\mu(d) = 2 \cdot 2 \cdots 2$ where the last product has $\omega(n)$ factors.