# HW1 - Solutions

1. (Davenport, pp.215-216, ex. 1.01)

(a) We want to show \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \). The claim clearly holds for \( n = 1 \) (i.e. \( 1 = \frac{2}{2} \)). Assume that it holds for \( n \) (that is, assume that \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \)). We want to prove the claim for \( n+1 \) (that is, \( \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2} \)). Indeed,

\[
\begin{align*}
\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2},
\end{align*}
\]

as required.

(b) We want to show \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \). The claim clearly holds for \( n = 1 \). Assume that it holds for \( n \). We want to prove the claim for \( n+1 \). Indeed,

\[
\begin{align*}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6},
\end{align*}
\]

as required.

(c) We want to show \( \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \). The claim clearly holds for \( n = 1 \). Assume that it holds for \( n \). We want to prove the claim for \( n+1 \). Indeed,

\[
\begin{align*}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
&= \frac{n^2(n+1)^2+4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4},
\end{align*}
\]

as required.

2. (Davenport, pp.215-216, ex.1.02)

(a) We want to show that \( F_n < \tau^n \) where \( \tau \) is the golden ratio, \( \frac{1+\sqrt{5}}{2} \). The first step is the check the statement for \( n = 1 \) and \( n = 2 \). Since
\( \sqrt{5} > 1, (1 + \sqrt{5}) > 2 \), and hence \( \tau > 1 = F_1 \). Similarly, \( \tau^2 > 1 = F_2 \), because \( \tau^2 = \tau + 1 \) (observe that \( \tau \) is the root of the second degree polynomial \( x^2 - x - 1 \); or you can verify it directly). The induction step is as follows: Assume that the statement holds for \( n - 1 \) and \( n \), i.e. \( F_{n-1} < \tau^{n-1} \) and \( F_n < \tau^n \). Then \( F_{n+1} = F_n + F_{n-1} < \tau^n + \tau^{n-1} = \tau^{n-1}(\tau + 1) = \tau^{n-1}\tau^2 = \tau^{n+1} \).

(b) Now, we want to prove that \( F_n = (\tau^n - \sigma^n)/\sqrt{5} \), where \( \sigma = -1/\tau = (1 - \sqrt{5})/2 \). Again, the first step is to check whether the statement is true for \( n = 1, 2 \). Indeed, \( (\tau - \sigma)/\sqrt{5} = 1 = F_1 \). Similarly, \( (\tau^2 - \sigma^2)/\sqrt{5} = ((\tau + 1) - (\sigma - 1))/\sqrt{5} = (\tau - \sigma)/\sqrt{5} = 1 = F_2 \), where we used the fact that \( \sigma^2 = \sigma + 1 \) (\( \sigma \) is the other root of \( x^2 - x - 1 \), or verify directly that \( \sigma^2 = \sigma + 1 \)). The induction step is as follows: assume that the statement is true for \( n = n - 1 \), i.e. \( F_n = (\tau^n - \sigma^n)/\sqrt{5} \) and \( F_{n-1} = (\tau^{n-1} - \sigma^{n-1})/\sqrt{5} \). Then, \( F_{n+1} = F_n + F_{n-1} = ((\tau^n - \sigma^n) - (\tau^{n-1} - \sigma^{n-1}))/\sqrt{5} = (\tau^{n-1}(\tau - 1) - \sigma^{n-1}(\sigma - 1))/\sqrt{5} = (\tau^{n+1} - \sigma^{n+1})/\sqrt{5} \), as claimed.

3. (Davenport, pp.215-216, ex. 1.03) Prime factorizations of given numbers are: \( 999 = 3^3 \cdot 37, 1001 = 7 \cdot 11 \cdot 13, 1729 = 7 \cdot 13 \cdot 19, 11111 = 41 \cdot 271, 65536 = 2^{16}, 6469693230 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \).

4. We will prove that \( n < 2^n \) for integers \( n \geq 1 \). For \( n = 1 \), the claim is obviously true. Now assume that \( n < 2^n \). Then \( n + 1 < 2^n + 1 < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1} \), as required.

5. We want to show that \( 1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2 = \sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^{n-1}n(n+1)/2 \). Easy to check for \( n = 1 \). Assume that it holds for \( n \). Then \( 1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2 + (-1)^n(n+1)^2 = (-1)^{n-1}n(n+1)/2 + (-1)^n(n+1)^2 = (-1)^n(n+1)(-n+2(n+1))/2 = (-1)^{n+1}(n+1)/2 \).

6. We claim that \( F_1 + F_3 + \ldots + F_{2n-1} = F_{2n} \) (once you calculate this sum for first few \( n \), you could immediately come up with this formula). Indeed, for \( n = 1 \), we have \( F_1 = F_2 = 1 \). Now, assume the formula is true for \( n \). Then, \( F_1 + F_3 + \ldots + F_{2n-1} + F_{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2} \), as desired.