# HW1 - Solutions

1. (Davenport, pp.215-216, ex. 1.01)

   (a) We want to show $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. The claim clearly holds for $n = 1$ (i.e. $1 = 2/2$). Assume that it holds for $n$ (that is, assume that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$). We want to prove the claim for $n+1$ (that is, $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$) . Indeed, $\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$.

   (b) We want to show $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. The claim clearly holds for $n = 1$. Assume that it holds for $n$. We want to prove the claim for $n+1$. Indeed,

   $$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)\left(n(2n+1)+6(n+1)\right)}{6} \\ &= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6},\end{aligned}$$

   as required.

   (c) We want to show $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. The claim clearly holds for $n = 1$. Assume that it holds for $n$. We want to prove the claim for $n+1$. Indeed,

   $$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{n^2(n+1)^2+4(n+1)^3}{4} = \frac{(n+1)^2\left(n^2+4n+4\right)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4},\end{aligned}$$

   as required.

2. (Davenport, pp.215-216, ex.1.02)

   (a) We want to show that $F_n < \tau^n$ where $\tau$ is the *golden ratio*, $(1+\sqrt{5})/2$. The first step is the check the statement for $n = 1$ and $n = 2$: Since

$\sqrt{5} > 1$, $(1+\sqrt{5}) > 2$, and hence $\tau > 1 = F_1$. Similarly, $\tau^2 > 1 = F_2$, because $\tau^2 = \tau + 1$ (observe that $\tau$ is the root of the second degree polynomial $x^2 - x - 1$; or you can verify it directly). The induction step is as follows: Assume that the statement holds for $n-1$ and $n$, i.e. $F_{n-1} < \tau^{n-1}$ and $F_n < \tau^n$. Then $F_{n+1} = F_n + F_{n-1} < \tau^n + \tau^{n-1} = \tau^{n-1}(\tau+1) = \tau^{n-1}\tau^2 = \tau^{n+1}$.

(b) Now, we want to prove that $F_n = (\tau^n - \sigma^n)/\sqrt{5}$, where $\sigma = -1/\tau = (1-\sqrt{5})/2$. Again, the first step is to check whether the statement is true for $n = 1, 2$. Indeed, $(\tau - \sigma)/\sqrt{5} = 1 = F_1$. Similarly, $(\tau^2 - \sigma^2)/\sqrt{5} = ((\tau+1) - (\sigma - 1))/\sqrt{5} = (\tau - \sigma)/\sqrt{5} = 1 = F_2$, where we used the fact that $\sigma^2 = \sigma + 1$ ($\sigma$ is the other root of $x^2 - x - 1$, or verify directly that $\sigma^2 = \sigma + 1$). The induction step is as follows: assume that the statement is true for $n$ and $n-1$, i.e. $F_n = (\tau^n - \sigma^n)/\sqrt{5}$ and $F_{n-1} = (\tau^{n-1} - \sigma^{n-1})/\sqrt{5}$. Then, $F_{n+1} = F_n + F_{n-1} = ((\tau^n - \sigma^n) - (\tau^{n-1} - \sigma^{n-1}))/\sqrt{5} = (\tau^{n-1}(\tau - 1) - \sigma^{n-1}(\sigma - 1))/\sqrt{5} = (\tau^{n+1} - \sigma^{n+1})/\sqrt{5}$, as claimed.

3. (Davenport, pp.215-216, ex. 1.03) Prime factorizations of given numbers are: $999 = 3^3 \cdot 37$, $1001 = 7 \cdot 11 \cdot 13$, $1729 = 7 \cdot 13 \cdot 19$, $11111 = 41 \cdot 271$, $65536 = 2^{16}$, $6469693230 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$.

4. We will prove that $n < 2^n$ for integers $n \geq 1$. For n=1, the claim is obviously true. Now assume that $n < 2^n$. Then $n + 1 < 2^n + 1 < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$, as required.

5. We want to show that $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2 = \sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^{n-1}\frac{n(n+1)}{2}$. Easy to check for $n = 1$. Assume that it holds for $n$. Then $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2 + (-1)^n(n+1)^2 = (-1)^{n-1}\frac{n(n+1)}{2} + (-1)^n(n+1)^2 = \frac{(-1)^{n-1}n(n+1) + (-1)^n 2(n+1)^2}{2} = \frac{(-1)^n(n+1)(-n+2(n+1))}{2} = (-1)^n\frac{(n+1)(n+2)}{2}$.

6. We claim that $F_1 + F_3 + \ldots + F_{2n-1} = F_{2n}$ (once you calculate this sum for first few $n$, you could immediately come up with this formula). Indeed, for $n = 1$, we have $F_1 = F_2 = 1$. Now, assume the formula is true for $n$. Then, $F_1 + F_3 + \ldots + F_{2n-1} + F_{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2}$, as desired.