SKETCH OF SOLUTIONS (HOMEWORK II)

3.1 a) If \(u = du'\) and \(v = dv'\) with \(d > 1\) then \(a = u^2 - v^2 = d(u'^2 - d(v')^2), \)
\(b = d(2u'v'),\) and \(c = d(d(u')^2 + d(v')^2)),\)
b) \(u = 5,\ v = 3\)
d) 1.- \(u\) and \(v\) have no common factor.
2.- \(u\) and \(v\) have different parities.
e) Suppose \((a = u^2 - v^2, b = 2uv, c = u^2 + v^2)\) satisfies the conditions in d), let \(p\) be a prime which is a common factor of \(a, b\) and \(c\), then \(p \mid (b + c) = (u + v)^2\) and \(p \mid (a + b) = (u - v)^2\) therefore \(p \mid (u + v)\) and \(p \mid (u - v)\). Therefore \(p \mid 2u\) and \(p \mid 2v\). Since \(u\) and \(v\) have different parities, \(c\) is odd, therefore \(p\) cannot be 2. The only possibility left is that \(p \mid a\) and \(p \mid b\), but \(u\) and \(v\) have no common factor. Therefore \(u\) and \(v\) have no common factor.
3.2 a) \(x = \frac{(m-1)^2 - 2}{1+m^2}, y = 1 - \frac{2m(m+1)}{1+m^2}\)
b) There should be at least one point with rational coordinates.
3.3 \(x = \frac{1+m^2}{1-m^2}, y = m(\frac{1+m^2}{1-m^2} + 1)\)
4.2 a) \((8, 8, 32), (18, 18, 108), (32, 32, 4098)\)
b) \((n^2A)^3 + (n^2B)^3 = n^6(A^3 + B^3) = n^6C^2 = n^3c^2\)
d) \(2a^3 = c^2 \Rightarrow 2 \mid c^2\) therefore \(c = 2t\). i.e. \(2a^3 = 4t^2\) which implies \(a^3 = 2t^2\), but then \(2 \mid a\). That is \(a = 2s\) and \(4s^3 = t^2\), therefore \(t = 2v\) and \(s^3 = v^2\). Substituting we get that the solution \((a, a, c)\) equals \((2(\sqrt{3}v)^2, 2(\sqrt{3}v)^2, 4(\sqrt{3}v)^3)\) Therefore \((2, 2, 4)\) is the only primitive solution with \(a = b\)
5.1 \(\gcd(12345, 67890) = 15), \gcd(54321, 9876) = 3\)