SKETCH OF SOLUTIONS (HOMEWORK I)

1.1 The next two triangle-square numbers are 1225 and 41616. Suppose we have a triangle-square number

\[ n^2 = \frac{m(m + 1)}{2} \]

There are two possibilities: Either \( m \) is even or \( m + 1 \) is even. Let us analyze the case where \( m \) is even (for example \( m = 8 \), where we get \( \frac{8(8+1)}{2} = 6^2 \)).

If \( m = 2t \) then we get

\[ n^2 = \frac{2t(2t + 1)}{2} = t(2t + 1) \]

Claim: The only possible common factor of \( t \) and \( 2t + 1 \) is 1:

Suppose \( d \mid t \) and \( d \mid (2t + 1) \) then \( d \mid (2t + 1) - t \ i.e. \ d \mid (t + 1) \), but then \( d \mid (t + 1) - t \ i.e. \ d \mid 1 \). Thus \( d = 1 \) By the claim we get that both \( t \) and \( 2t + 1 \) are squares (since their product is a square), that is, \( t = s^2 \) and

\[ 2t + 1 = 2s^2 + 1 = u^2 \]

Notice that if we have a solution to the last equation we get a solution to our problem, indeed, if \( 2s^2 + 1 = u^2 \) then \( s^2(2s^2 + 1) = s^2u^2 = (su)^2 \) therefore \( \frac{2s^2(2s^2+1)}{2} = (su)^2 \), defining \( m = 2s^2 \) and \( n = su^2 \) we get a triangle-square number.

Now we will show that there are infinitely many triangle-square numbers by exhibiting infinitely many solutions to the equation

\[ 2s^2 + 1 = u^2 \]

First let’s find a single solution. Substituting the appropriate values for the example \( m = 8 \) we get: \( m = 8 = 2t \) therefore \( t = 4 \), also \( t = 4 = s^2 \) therefore \( s = 2 \), finally we get the solution \( 2s^2 + 1 = 2(4) + 1 = 9 = u^2 \) therefore \( u = 3 \) Now let’s produce infinitely many solutions from a given solution:

Claim: If \( s \) and \( u \) satisfy

\[ 2s^2 + 1 = u^2 \]

then \( \bar{s} = 2su \) and \( \bar{u} = (2u^2 - 1) \) satisfy

\[ 2\bar{s}^2 + 1 = \bar{u}^2 \]

Proof

\[ 2s^2 + 1 = 2(2su)^2 + 1 = 2(4s^2u^2) + 1 = 4(2s^2u^2) + 1 \]

Now we use the hypothesis

\[ 2s^2 + 1 = u^2 \]

and we get:

\[ 2\bar{s}^2 + 1 = 4(2s^2u^2) + 1 = 4(u^2 - 1)u^2 + 1 = 4u^4 - 4u^2 + 1 = (2u^2 - 1)^2 = \bar{u}^2 \]

A complete solution to the problem is given in chapter 28.
1.3 Suppose $p, p + 2$ and $p + 4$ are primes with $p \geq 3$. If $p = 3n + 1$ then $p + 2 = 3n + 3 = 3(n + 1)$ which is divisible by 3! If $p = 3n + 2$ then $p + 4 = 3n + 6 = 3(n + 2)$ which is divisible by 3! Therefore $p = 3n$, that is possible only if $n = 1$ therefore the only prime triplet is 3, 5, 7.

1.4 Notice that $n^2 - a^2 = (n - a)(n + a)$.

2.1 a) The only squares modulo 3 are 0 and 1 and $1 + 1 = 2$ therefore, at least one of $a$ and $b$ must be 0 modulo 3.

b) The only squares modulo 5 are 0, 1 and 4, if neither $a$ nor $b$ are divisible by five, the only possibility left is that one of them leaves a residue of 1 and the other a residue of 4, but then $c$ leaves a residue of zero. Therefore at least one of $a, b, c$ is divisible by 5.

2.3 a) Any odd number $2n + 1$: $a = st = (2n + 1)1$ (if $n = 0$ we take $b = 0$ and $c = 1$ otherwise we use the theorem).

b) Any multiple of 4:

$$b = \frac{s^2 - t^2}{2}$$

with $s = 2n + 1$, $t = 2m + 1$ and $n > m$. Therefore $b = \frac{4[n(n+1)-m(m+1)]}{2}$ since both $n(n + 1)$ and $m(m + 1)$ are even we get $b = \frac{4k}{1} = 4k$. On the other hand, we can express any multiple of 4 as in the expression for $b$:

$$4k = \frac{(2k + 1)^2 - (2k - 1)^2}{2}$$

2.4 The following is a list with 8 different primitive Pythagorean triples with $c = 32045$:

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<th>a</th>
<th>b</th>
<th>c</th>
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