SKETCH OF SOLUTIONS (HOMEWORK I)

1.1 The next two triangle-square numbers are 1225 and 41616. Suppose we have a triangle-square number

$$n^2 = \frac{m(m+1)}{2}$$

There are two possibilities: Either m is even or $m+1$ is even. Let us analyze the case where m is even (for example $m = 8$, where we get $\frac{8(8+1)}{2} = 6^2$). If $m = 2t$ then we get

$$n^2 = \frac{2t(2t+1)}{2} = t(2t+1)$$

Claim: The only possible common factor of t and $2t + 1$ is 1:
Suppose $d \mid t$ *and* $d \mid (2t+1)$ *then* $d \mid (2t+1) - t$ *i.e.* $d \mid (t+1)$*, but then* $d \mid (t+1) - t$ *i.e.* $d \mid 1$. *Thus* $d = 1$ By the claim we get that both t and $2t + 1$ are squares (since their product is a square), that is, $t = s^2$ and

$$2t + 1 = 2s^2 + 1 = u^2$$

Notice that if we have a solution to the last equation we get a solution to our problem, indeed, if $2s^2 + 1 = u^2$ then $s^2(2s^2 + 1) = s^2u^2 = (su)^2$ therefore $\frac{2s^2(2s^2+1)}{2} = (su)^2$, defining $m = 2s^2$ and $n = su^2$ we get a triangle-square number.
Now we will show that there are infinitely many triangle-square numbers by exhibiting infinitely many solutions to the equation

$$2s^2 + 1 = u^2$$

First let's find a single solution. Substituting the appropriate values for the example $m = 8$ we get: $m = 8 = 2t$ therefore $t = 4$, also $t = 4 = s^2$ therefore $s = 2$, finally we get the solution $2s^2 + 1 = 2(4) + 1 = 9 = u^2$ therefore $u = 3$ Now let's produce infinitely many solutions from a given solution:
Claim: If s and u satisfy

$$2s^2 + 1 = u^2$$

then $\bar{s} = 2su$ and $\bar{u} = (2u^2 - 1)$ satisfy

$$2\bar{s}^2 + 1 = \bar{u}^2$$

Proof

$$2\bar{s}^2 + 1 = 2(2su)^2 + 1 = 2(4s^2u^2) + 1 = 4(2s^2u^2) + 1$$

Now we use the hypothesis

$$2s^2 + 1 = u^2$$

and we get:

$$2\bar{s}^2 + 1 = 4(2s^2u^2) + 1 = 4(u^2 - 1)u^2 + 1 = 4u^4 - 4u^2 + 1 = (2u^2 - 1)^2 = \bar{u}^2$$

A complete solution to the problem is given in chapter 28.

1.3 Suppose p,$p+2$ and $p+4$ are primes with $p \geq 3$. If $p = 3n+1$ then $p+2 = 3n+3 = 3(n+1)$ which is divisible by 3! If $p = 3n+2$ then $p+4 = 3n+6 = 3(n+2)$ which is divisible by 3! Therefore $p = 3n$, that is possible only if $n = 1$ therefore the only prime triplet is $3, 5, 7$

1.4 Notice that $n^2 - a^2 = (n-a)(n+a)$

2.1 a) The only squares modulo 3 are 0 and 1 and $1+1 = 2$ therefore, at least one of a and b must be 0 modulo 3

b) The only squares modulo 5 are 0, 1 and 4, if neither a nor b are divisible by five, the only possibility left is that one of them leaves a residue of 1 and the other a residue of 4, but then c leaves a residue of zero. Therefore at least one of a, b, c is divisible by 5

2.3 a) Any odd number $2n+1$: $a = st = (2n+1)1$ (if $n = 0$ we take $b = 0$ and $c = 1$ otherwise we use the theorem)

b) Any multiple of 4:

$b = \frac{s^2-t^2}{2}$ with $s = 2n+1$, $t = 2m+1$ and $n > m$. Therefore $b = \frac{4[n(n+1)-m(m+1)]}{2}$ since both $n(n+1)$ and $m(m+1)$ are even we get $b = \frac{8k}{2} = 4k$ On the other hand, we can express any multiple of 4 as in the expression for b:

$$4k = \frac{(2k+1)^2 - (2k-1)^2}{2}$$

2.4 The following is a list with 8 different primitive Pythagorean triples with $c = 32045$:

a	b	c
2277	31964	32045
8283	30956	32045
17253	27004	32045
21093	24124	32045
23067	22244	32045
27813	15916	32045
31323	6764	32045
32037	716	32045