# The Peano Axioms

The Peano axioms define the natural numbers, often denoted as $\mathbb{N}$. They were introduced in 1889 by Giuseppe Peano.

(1) 0 is a natural number.

(2) For every natural number $n$, the **successor** of $n$ is also a natural number. We denote the successor of $n$ by $S(n)$.

(3) For every natural number $n$, $S(n) = 0$ is false.

(4) For all natural numbers $m$ and $n$, $S(m) = S(n)$ if and only if $m = n$.

(5) If $K$ is a set of natural numbers such that
- 0 is in $K$, and
- for every natural number $n$, if $n \in K$ then $S(n) \in K$.

then $K$ contains every natural number.

These axioms formalize the idea of counting. Intuitively, to count a collection of things, we start with not having anything counted (i.e. 0), which is Property (1). Property (2) is a way of saying that after every number $n$, there is a next number $S(n)$ (which we think of as $n+1$, but addition isn't directly defined by the axioms).

Property (3) tells us that there is no natural number "before" zero, and Property (4) tells us that each number has exactly one successor.

Property (5) is perhaps the hardest one to understand, but in some very definite sense, it tells us that $\mathbb{N}$ corresponds exactly to things that we can count, given an arbitrary long amount of time. We have to start counting (ie, $0, S(0) = 1, S(1) = 2, \ldots$) and at each step, we are always able to count one more thing. It tells us that every natural number can be "reached" by enough applications of the successor function $S$.

This is useful in establishing properties about $\mathbb{N}$, and goes by the name of "mathematical induction" (or often, just "induction"). Despite the similarities in the name, is quite distinct from the idea of inductive reasoning, in which one draws (possibly false) conclusions from a number of examples. In fact, mathematical induction is a type of deductive reasoning. English can be weird like that.

We should remark that some versions of the Peano Axioms begin with the number 1 rather than 0, and some authors refer to the set defined about as the "whole numbers", and use the term "natural number" to refer to the nonzero whole numbers. In fact, Peano's original formulation used 1 as the "first" natural number.

As an example of induction (assuming we know the rules of arithmetic), we can show that

**Theorem 1.** *For any natural number n, the sum of the first n odd naturals is $n^2$. That is,*

$$1 + 3 + 5 + \ldots + (2n+1) = n^2.$$

*Proof.* To put this in the language of property (5), we let $K$ be the set of natural numbers for which this is true, that is

$$K = \left\{ n \in \mathbb{N} \mid 1 + 3 + 5 + \ldots + (2n-1) = n^2 \right\}.$$

We want to show that $K = \mathbb{N}$.

First, we establish the "base case", that is, that $0 \in K$. If we add zero odd integers, the sum is $0 = 0^2$. So that holds. We don't actually need to establish the next few cases, but let's do it anyway to get a feel for what is happening.

$1 \in K$, since $1 = 1^2$, and $2 \in K$, since $1+3 = 4 = 2^2$. Let's look at the second one again by rewriting 3 as $(2+1)$. Then we have

$$1+3 = 1+(2+1) = (1+1)^2 = 2^2.$$

Let's see what is going on when we add the first three and the first four odds:

$$1+3+5 = 4+5 = 2^2+(2\cdot 2+1) = (2+1)^2 = 3^2$$
$$1+3+5+7 = 9+7 = 3^2+(2\cdot 3+1) = (3+1)^2 = 4^2$$

Perhaps the pattern is clear now. We can write the sum of the first $k+1$ odd numbers in terms of the sum of the first $k$ odds and the next one. Then if we know that the first $k$ odds add up to $k^2$, we can regroup and see the whole sum as a perfect square. More formally, if we know that $k \in K$ (that is, $1+3+5+\ldots+(2k-1) = k^2$), we want to show that $k+1 \in K$, that is,

$$1+3+5+\ldots+(2k-1)+(2(k-1)+1) = (k+1)^2.$$

But, we already know that the sum of the first $k$ odds is $k^2$, so we really just want to show that

$$k^2+(2(k-1)+1) = (k+1)^2,$$

which is apparent since the left-hand side is just $k^2+2k-2+1$, certainly equal to the right-hand side $k^2+2k+1$. Since we have shown that $0 \in K$ and $n+1 \in K$ whenever $n \in K$, property (5) tells us that $K = \mathbb{N}$, that is, the assertion holds for every natural number. □

Another immediate consequence of the Peano axioms is that we have a built-in ordering on $\mathbb{N}$. That is, we can define "$<$" by

**Definition 2.** *For two natural numbers m and n, we say that m* **is less than** *n (or $m < n$) if*

$$S(m) = n \quad \text{or} \quad S(m) < n.$$

**Definition 3.** *For two natural numbers m and n, we say that n* **is greater than** *m (or $n > m$) if and only if $m < n$.*

From this, the "Trichotomy property" follows very quickly.

**Proposition 4.** *If m and n are two natural numbers, then exactly one of the following holds.*

$$\text{Either} \quad m < n \quad \text{or} \quad m = n \quad \text{or} \quad m > n.$$