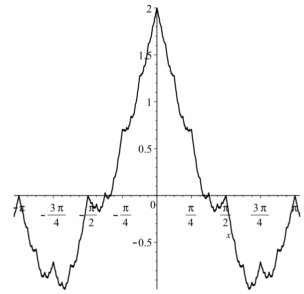
MAT513 Homework 9

Due Wednesday, April 26

Problems marked with a * are optional/extra credit. However, please at least consider them.



1. Show that $w(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$ is a continuous function on all of $\mathbb{R}$. (The Weierstrauss M-Test will probably be helpful).
This function is closely related to Weierstrauss's continuous, nowhwere differentiable function.

2. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and suppose also that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Show that if $L > 0$, then the series converges for all $x \in (-1/L, 1/L)$, and if $L = 0$, the series converges for all $x \in \mathbb{R}$.

You might want to refer to problem 3 on homework 4 about the ratio test.

3. Show that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2 \quad \text{and that} \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6.$$

Hint: Start with the series for $1/(1-x)$.

4. Consider the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

We saw that $g'(0) = 0$, and I claimed (but did not demonstrate) that $g^{(n)}(0) = 0$ for $n \in \mathbb{N}$. Consequently, the Taylor series for $g(x)$ is identically zero, but $g(x) \neq 0$ for $x \neq 0$. This function is infinitely differentiable, but not analytic.

(a) Compute $g'(x)$ for $x \neq 0$, and then use the definition of the derivative to compute $g''(0)$.

(b) For $x \neq 0$, $g''(x) = (-6x^{-4} + 4x^{-6})e^{-1/x^2}$ and $g'''(x) = (24x^{-5} - 36x^{-7} + 8x^{-9})e^{-1/x^2}$. Give a general description of $g^{(n)}(x)$ for $x \neq 0$ (it doesn't have to be an exact formula, but you need to fully justify what you claim), and then use that description to prove that $g^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

5. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ both converge for $x \in (-R, R)$ for some $R > 0$. Suppose also that $f'(x) = g(x)$, $g'(x) = -f(x)$, $f(0) = 1$, and $g(0) = 0$. Determine the Taylor series for $f(x)$. Is this a familiar function? What is the radius of convergence of the resulting series (that is, what is the largest value of R)?

6. Prove that every continuous function $g\colon [a,b] \to \mathbb{R}$ can be uniformly approximated by a polygonal function. That is, given $\varepsilon > 0$, there is a function $\phi_\varepsilon(x)$ so that $|g(x) - \phi_\varepsilon(x)| < \varepsilon$ for $x \in [a,b]$; the function $\phi_\varepsilon(x)$ is continuous and $[a,b]$ can be subdivided into n smaller intervals $[a_i, a_{i+1}]$ on which $\phi_\varepsilon(x) = m_i(x - x_i) + c_i$ for x in each of these intervals.
(Hint: consider what $g(x)$ continuous tells you.)

***7**. An outline of a proof that e is irrational is below. Fill in the details.

(a) First, observe that e^x is a strictly increasing function, that is, if $y > x$, then $e^y > e^x$. (This follows from the Mean Value Theorem and the fact that the derivative of e^x is positive at every x.)

(b) Using the Taylor series for e^x and the estimate $e < 3$, show that for all $n \in \mathbb{N}$,

$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right) < \frac{3}{(n+1)!}.$$

(c) Now suppose e were rational. Then there are integers p and q so that $e = p/q$. Let $n > \max(q, 3)$, and replace e by p/q in the above inequality to show there must be an integer between 0 and $3/4$. Hence, e cannot be rational.

***8**. Consider this extended version of the second derivative test. Use Taylor's theorem to prove that it works.

Let $f\colon A \to \mathbb{R}$ be analytic in an interval around a point $c \in A$. (That is, the Taylor series converges to $f(x)$ near c.) Suppose that there is an integer $n \geq 2$ so that $f^{(n)}(c) \neq 0$ but $f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$. Then:

- If n is even and $f^{(n)}(c) < 0$, then f has a local maximimum at $x = c$.
- If n is even and $f^{(n)}(c) > 0$, then f has a local minimum at $x = c$.
- If n is odd, then f has neither a local maximum or a local minimum at $x = c$.

9. Consider the following question seen recently:

> I'm a high school math teacher teaching an introductory calculus course, and I'm having a problem teaching one particular student about the geometric definition of an integral.
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> The intuition is that it's the "area under the curve" and all but one of my students accept that this implicitly means "area under the curve down to the x-axis," but one student is hung up on thinking that "area under the curve" extends all the way down to $y = -\infty$.
>
> I tried giving him the following proof:
>
> > Suppose $\int_0^1 f(x)\ dx$ is the area under $f(x)$ all the way down to $y = -\infty$. It is clear visually that $\int_0^1 f(x) - g(x)\ dx$ is the area between $f(x)$ and $g(x)$. Now suppose $g(x) = 0$. Then $\int_0^1 f(x) - 0\ dx = \int_0^1 f(x)\ dx$ is the area between $f(x)$ and $y = 0$, contradicting the initial assumption.
>
> He seems to be unconvinced by this. What should I tell him?

Write a paragraph or twenty in response to the teacher. You might want to mention other "physical" meanings of the integral, or not.