MAT513 Homework 8

Due Wednesday, April 12

Problems marked with a * are optional/extra credit. However, please at least consider them.

1. Let $f_n(x) = \dfrac{nx}{1+nx^2}$.

(a) Find the pointwise limit of $\{f_n\}$ for $x \in (0,\infty)$.

(b) Is the convergence uniform on $(0,\infty)$? Is it uniform on $(0,1)$? Is it uniform on $(1,\infty)$?

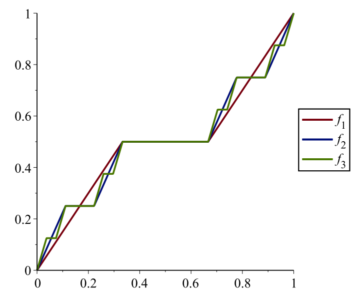
2. Let $f_n(x) = f(x+\frac{1}{n})$.

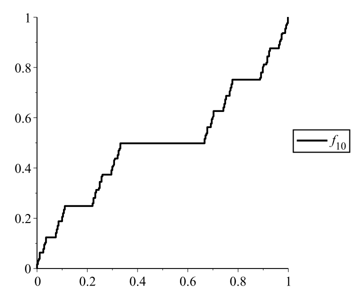
(a) If f is uniformly continuous on $\mathbb{R}$, show that $f_n \to f$ uniformly.

(b) Give an example of f which is continuous (but not uniformly continuous) where f_n does not converge to f uniformly.

3. Give an example of sequences $\{f_n\}$ and $\{g_n\}$ which converge uniformly, but the sequence of products $\{f_n g_n\}$ does not converge uniformly.

***4.** **(The Cantor function)** Recall our earlier discussion of the middle-thirds Cantor Set $\mathcal{C}$.





For $x \in [0,1]$, define $f_0(x) = x$ and for $n > 0$, let

$$f_n(x) = \begin{cases} \frac{1}{2} f_{n-1}(3x) & \text{for } 0 \le x \le \frac{1}{3} \\ \frac{1}{2} & \text{for } \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{2} f_{n-1}(3x-2) + \frac{1}{2} & \text{for } \frac{2}{3} \le x \le 1 \end{cases}.$$

Show that $\{f_n\}$ converges uniformly to a function f on $[0,1]$. Then show that f is a continuous, increasing function on $[0,1]$ with $f(0)=0$, $f(1)=1$, and satisfying $f'(x)=0$ for all x in $[0,1] \setminus \mathcal{C}$.

Since the "length" of the Cantor Set $\mathcal{C}$ is 0, this function f manages to increase from 0 to 1 while remaining constant on a set with "length 1". The graph of f (and of similar functions) is sometimes called a "devil's staircase".

5. Let $g_n(x) = x^n/n$ for $x \in [0,1]$.

(a) Show that $\{g_n\}$ converges uniformly on $[0,1]$ and find $g = \lim g_n$. Then show that $g(x)$ is differentiable and compute $g'(x)$.

(b) Now show that $\{g_n'(x)\}$ converges on $[0,1]$; is the convergence of g_n' uniform on $[0,1]$? Let $h = \lim g_n'$ and compare h and g'.

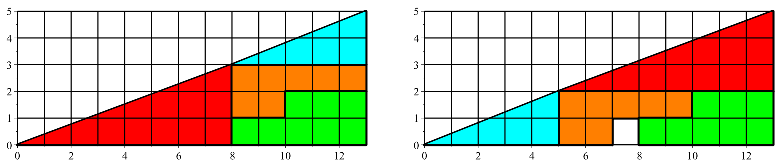
6. Is it possible to have $f_n \to f$ uniformly on $\mathbb{R}$, with each function f_n continuous and nowhere differentiable, but so that the limit function f is differentiable at every $x \in \mathbb{R}$? If so, give an explicit example (you can base your f_n on the function discussed in section 5.4 of the text and in class on 3/29, or another one). If not, explain why not.

7. Consider the picture at right below, a "proof without words" of something. What is being proven?

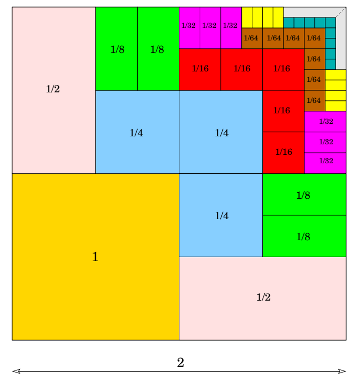
First, write an explanation of what is being demonstrated by this image in a way that can be understood by a student who knows something (not a lot) about infinite series.

Then, discuss whether you think this constitutes a convincing proof. Even if not, is this image helpful? Explain.

You might want to consider the image below, a "standard proof that $65/2 = 63/2$", as part of your discussion.



(See also Wikipedia: "Missing square puzzle").



Roger B. Nelsen,
Mathematics Magazine **62**
(Dec. 1989), pp.332–333