MAT513 Homework 7
Due Wednesday, April 5

1. Let $Q(x) = \frac{1}{x}$.

(a) Using the definition of the derivative, show that $Q'(x) = -\frac{1}{x^2}$.

(b) Provide a proof of the quotient rule in two ways: First, by algebraic manipulation of the difference quotient in the limit

$$\lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x-c},$$

and then again by combining the chain rule, the product rule, and the first part of this problem.

2. The **power rule** says that if $r \in \mathbb{R}$ and $P(x) = x^r$, then $P'(x) = rx^{r-1}$. You can assume the power rule in this problem (and elsewhere except in problem **1**.) Let

$$f_r(x) = \begin{cases} x^r & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

(a) For what r is $f_r(x)$ continuous at zero?

(b) For what r is $f_r(x)$ differentiable at zero?

(c) For what r is $f_r(x)$ differentiable at zero with a continuous derivative?

3. By analogy with the definition of uniform continuity, let's say that a function $f\colon A \to \mathbb{R}$ is **uniformly differentiable** on A if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$\left| \frac{f(x) - f(y)}{x-y} - f'(y) \right| < \varepsilon \quad \text{whenever} \quad 0 < |x-y| < \delta \text{ with } x, y \in A.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? What about $g(x) = x^3$?

(b) Show that if a function f is uniformly differentiable on an interval A, then the derivative of f must be continuous on A.

4. Let $f\colon [a,b] \to \mathbb{R}$ be a one-to-one function, and let $B = f([a,b])$. Then there is an inverse function $f^{-1}\colon B \to [a,b]$ given by $f^{-1}(y) = x$ where $f(x) = y$. In the previous homework (problem 5), you were asked to show that if f is continuous, then so is f^{-1}.

Assume f is differentiable on $[a,b]$ with $f'(x) \neq 0$ for every $x \in [a,b]$. Show that f^{-1} is differentiable on B with $\left(f^{-1}\right)'(y) = 1/f'(x)$ where $y = f(x)$.

5. Suppose f is differentiable on an interval A. Prove that if $f'(x) \neq 0$ on A, then f must be one-to-one on A. Give an example that shows the converse does not always hold.

6. Let f be twice differentiable on an open interval containing the point c, and suppose that f'' is continuous at c.

(a) Show that $f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h}$.

(b) Show that $f''(c) = \lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}$.

Hint: Write f'' in terms of f' via the Mean Value Theorem, then use the previous part.

7. In 2005, police in Scotland installed cameras at certain points along the A77 roadway to record license numbers and automatically calculate the average speed of individual cars between certain points along the road, then automatically issue speeding tickets to drivers whose average speed exceeded the limit. Some drivers objected that merely recording their positions at certain times was no proof that they were speeding at any given moment, especially since they slowed down when passing the cameras.

Write a paragraph or so responding to the claim, probably with some appeal to the mean value theorem.