MAT513 Homework 6

Due Wednesday, March 29

Problems marked with a * are optional/extra credit. However, please at least consider them.

1. Observe that if a and b are real numbers, then we can define $\max(a,b) = \dfrac{(a+b)+|a-b|}{2}$; this can readily be extended to a finite set of numbers $\{a_1, a_2, \dots, a_n\}$ via

$$\max\{a_1, a_2, \dots, a_n\} = \max(a_1, \max\{a_2, a_3, \dots, a_n\}).$$

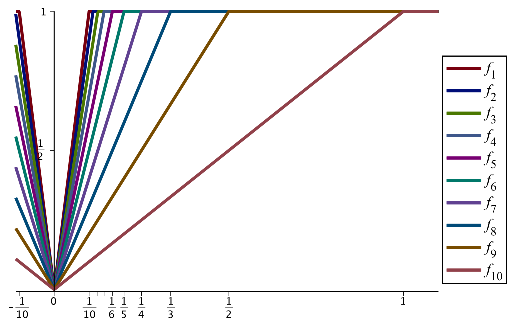
(a) Show that if $f_1, f_2, \dots, f_n$ are continuous, then $g(x) = \max(f_1(x), f_2(x), \dots, f_n(x))$ is also continuous.

(b) For each positive integer n, define

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n \end{cases}$$

For each n, $f_n \colon \mathbb{R} \to \mathbb{R}$ is continuous.

Write an explicit formula for the function
$h(x) = \sup\{f_1(x), f_2(x), f_3(x), \dots\}$.
Is $h(x)$ continuous?



2. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is continuous.

(a) Prove that if $B \subseteq \mathbb{R}$ is open, then $f^{-1}(B)$ is also open, where $f^{-1}(B) = \{x \mid f(x) \in B\}$.

(b) Suppose that the above property holds for every open set B, that is, for every open set $B \subseteq \mathbb{R}$, we have $g^{-1}(B)$ open. Is $g(x)$ necessarily a continuous function? Prove or give a counterexample.

3. Suppose that $f \colon [a,b] \to [a,b]$ is continuous. Prove that f has a **fixed point**; that is, that there is a $c \in [a,b]$ so that $f(c) = c$.

***4**. A function $f \colon A \to \mathbb{R}$ is called **Lipshitz** if there exists a real number $M > 0$ for which

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \qquad \text{for all } x \neq y \in A.$$

Geometrically, this means that there is a uniform bound on the slopes of lines drawn between any two points on the graph of f.

Show that if $f \colon A \to \mathbb{R}$ is Lipshitz, then it is uniformly continuous on A.
Does the converse hold? That is, must every uniformly continuous function also be Lipshitz? (Answer, no, or why else would it have its own name? Think about $g(x) = \sqrt{x}$.)

5. Suppose that $f \colon A \to \mathbb{R}$ is a one-to-one function. Let $B = f(A)$. Then we can define the **inverse function** $f^{-1} \colon B \to A$ by $f^{-1}(y) = x$ where $y = f(x)$.

Show that if f is continuous and one-to-one on an interval $[a,b]$, then f^{-1} is also continuous.

6. Assume that the temperature $T(x)$ of a point x on the equator of the Earth is a continuous function. As a corollary to the Intermediate Value Theorem, at every moment there is a point x on the equator with the property that its antipodal point (the point $-x$ which is immediately opposite it on a line through the center of the Earth) has exactly the same tempertature, that is $T(x) = T(-x)$.

Write a paragraph or two explaining this in a way that it can be understood by a high school student.