MAT513 Homework 4

Due Wednesday, March 1

Problems marked with a * are optional/extra credit. However, please at least consider them.

1. Provide a proof of the **Comparison Test for Series** (p. 72 of the text), either using the Cauchy Criterion for Series or the Monotone Convergence Theorem.

2. An invented defintion: A series **subverges** if the sequence of partial sums $\{s_n\}$ has a convergent subsequence. For each of the statements below, decide if it is true or false; justify your answer with either a sketch of a proof or a counter example.

(a) If $\{a_n\}$ is bounded, then $\sum a_n$ subverges.

(b) Every convergent series is subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ also subverges.

(d) If $\sum a_n$ subverges, then $\{a_n\}$ has a convergent subsequence.

3. (**Ratio Test**) Given a series $\sum a_n$ with $a_n \neq 0$, the ratio test says that if $\{a_n\}$ satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r \quad \text{with } r < 1,$$

then $\sum a_n$ converges absolutely.

Prove the ratio test.

(Hint: Let R satisfy $r < R < 1$ and show that for n sufficiently large, $|a_{n+1}| \leq |a_n|R$. Then show $|a_N| \sum R^n$ converges (for a certain, fixed value of N), and use that to show $\sum |a_n|$ converges.)

***4**. (a) Let $\{a_n\}$ satisfy $a_n > 0$ for all n and $\lim na_n = L$ with $L > 0$. Prove $\sum a_n$ diverges.

(b) Let $\{b_n\}$ satisfy $b_n > 0$ for all n and $\lim n^2 b_n = L$ with $L > 0$. Prove $\sum b_n$ converges.

5. Let $A = \left\{ \dfrac{(-1)^n n}{n+1} \;\middle|\; n \in \mathbb{N} \text{ with } n > 1 \right\}$. Answer the following questions about A, justifying your answers fully.

(a) Is A an open set?

(b) Is A a closed set?

(c) Does A contain any isolated points? (If so, identify them.)

(d) What is $\bar{A}$?

***6**. Theorem 3.2.12 says: *For any $A \subseteq \mathbb{R}$, the closure $\bar{A}$ is a closed set, and is the smallest closed set which contains A.*

Let $L = \{ x \mid x \textit{ is a limit point of } A \}$. Recall that $\bar{A} = A \cup L$. A partial proof of the theorem is in the text (p. 92); you should complete the proof by showing that if y is a limit point of L, then $y \in L$ (that is, show that the set of limit points is closed).

7. Three types of sets are described below; only one is possible and the other two cannot be realized. Identify which is which. Provide an explicit example for the one which is possible, and explain why the other two cannot exist.

- $A \subseteq [0,1]$ is countably infinite, but has no limit points.
- $B \subseteq [0,1]$ is countably infinite, and contains no isolated points.
- $C \subseteq [0,1]$ contains an uncountable number of isolated points.

***8**. A notion dual to the closure of a set is the interior of a set. Specifically, given a set E, the **interior of E** is denoted by $\mathring{E}$ and is defined as

$$\mathring{E} = \{\, x \in E \mid \textit{there exists a neighborhood } V_\varepsilon(x) \subseteq E \,\}.$$

There is a symmetry between many properties regarding closure and the interior of sets.

(a) We know that a set E is closed if and only if $E = \overline{E}$. Show that a set U is open if and only if $U = \mathring{U}$.

(b) Recall that E^c denotes the complement of E, that is, $E^c = \mathbb{R} \setminus E$.
Show that $\overline{E}^c$ is the interior of E^c. Also show that $(\mathring{E})^c = \overline{E^c}$.

9. Below are several statements about compact sets; some are true and some are not. Prove the true ones, and give a counterexample for the false ones.

(a) Let K_n be a compact set for each $n \in \mathbb{N}$. Then $K = \displaystyle\bigcup_{n=0}^{\infty} K_n$ must be compact.

(b) Let F_λ be a compact set for each $\lambda \in \Lambda$. Then $F = \displaystyle\bigcap_{\lambda \in \Lambda} F_\lambda$ must be compact.

(c) Let A be an arbitrary subset of $\mathbb{R}$, and let K be compact. Then $A \cap K$ is compact.

(d) Let K be compact and F be closed. Then $K \setminus F = \{\, x \in K \mid x \notin F \,\}$ is open.

***10**. Let $\mathcal{C}$ denote the middle-thirds Cantor set. The goal of this exercise is to show the surprising result that

$$\mathcal{C} + \mathcal{C} = \{\, x + y \mid x, y \in \mathcal{C} \,\} = [0,2].$$

That is, any real number z with $0 \le z \le 2$ can be written as $x + y$, where x and y are in $\mathcal{C}$. (The other direction is obvious: since $\mathcal{C} \subset [0,1]$, certainly $0 \le x + y \le 2$.)

Prove this result.[†]

11. Read the excerpt consisting of pp. 239–249 from chapter 8 of Outliers by Malcom Gladwell at `www.math.stonybrook.edu/~scott/mat513.spr17/HW/gladwell.pdf`

Then write a paragraph or two in response. Is this relevant to teaching mathematics? How so (or why not)?

[†] Here is a sketch of one way to prove this. (There are other ways; feel free to use a different one.)

Recall that $\mathcal{C}$ is constructed as $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \cap \cdots$, where $\mathcal{C}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $\mathcal{C}_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and $\mathcal{C}_{n+1}$ is obtained from $\mathcal{C}_n$ by removing the 2^n open intervals forming the middle-third of each of the closed intervals of the form $[\frac{i}{3^n}, \frac{i+1}{3^n}]$ that make up $\mathcal{C}_n$.

Given any $z \in [0,2]$, observe that there are $x_1 \in \mathcal{C}_1$ and $y_1 \in \mathcal{C}_1$ with $x_1 + y_1 = z$. Show that for any $n \in \mathbb{N}$, there are numbers $x_n \in \mathcal{C}_n$ and $y_n \in \mathcal{C}_n$ with $x_n + y_n = z$.

Now, even though the sequences $\{x_n\}$ and $\{y_n\}$ may not converge, they can be used to construct points $x \in \mathcal{C}$ and $y \in \mathcal{C}$ so that $x + y = z$.

It may be helpful to note that $\mathcal{C}$ is compact: $\mathcal{C}$ is bounded, and since it is the complement of an open set, $\mathcal{C}$ is closed.