MAT513 Homework 3

Due Wednesday, February 22

1. (**Cesàro Means**) Given a sequence $\{x_n\}$, define a new sequence whose terms are the arithmetic mean of the first n terms. That is, let

$$y_n = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

(a) Show that if $\{x_n\}$ converges to a limit L, then the sequence of averages $\{y_n\}$ also converges to the same limit.

(b) Give an example where the sequence of averages $\{y_n\}$ converges but the original sequence $\{x_n\}$ does not.

2. Let $\{x_n\}$ be given by $\begin{cases} x_1 = 3 \\ x_{n+1} = \frac{1}{4-x_n} \end{cases}$

(a) Prove that $\{x_n\}$ converges. (Hint: is it monotone?)

(b) Use that fact that $\{x_n\}$ and $\{x_{n+1}\}$ converge to the same limit to calculate $\lim x_n$.

3. Given a bounded sequence a_n, let $x_n = \sup\{\, a_k \mid k \geq n \,\}$.

(a) Prove that $\{x_n\}$ always converges.
This limit is called the **limit superior** of $\{a_n\}$, denoted by $\limsup a_n$. Similarly, we can define $\liminf a_n$ as the limit of the sequence given by $y_n = \inf\{\, a_k \mid k \geq n \,\}$.

(b) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence $\{a_n\}$. Give an example of a bounded sequence $\{a_n\}$ where $\liminf a_n < \limsup a_n$.

(c) Prove that $\liminf a_n = \limsup a_n$ if and only if $\{a_n\}$ converges. (In that case, all three limits are the same.)

4. For a sequence $\{x_n\}$, we say x_m is a **peak term** if it is no smaller than any following terms, that is, if $x_m \geq x_n$ for all $n > m$.

(a) Give examples of sequences with zero, one, and two peak terms. Also, give an example of a sequence which is not monotone but with infinitely many peak terms.

(b) Using the concept of peak terms, show that every sequence contains a monotone subsequence.

(c) Explain why the Bolzano-Weierstrauss Theorem follows from the result in prob **4**.b.

5. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences. In each part, decide whether $\{c_n\}$ is Cauchy or not, and fully justify your answer.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = \lfloor a_n \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

6. Consider the following (invented) definition:
A sequence s_n is **pseudo-Cauchy** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that whenever $n \geq N$, we have $|s_{n+1} - s_n| < \varepsilon$.

Below are two propositions, of which one is true and one is false. Decide which is which, and provide a proof of the true one and a counterexample for the false one.

(i) Every pseudo-Cauchy sequence is bounded.

(ii) If $\{x_n\}$ and $\{y_n\}$ are pseudo-Cauchy, then $\{x_n + y_n\}$ is also pseudo-Cauchy.

7. Another invented definition:
A sequence $\{a_n\}$ is **quasi-increasing** if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ so that whenever $n > m \geq N$ it follows that $a_n > a_m - \varepsilon$.

(a) Give an example of a sequence which is quasi-increasing but is not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? That is, suppose $\{a_n\}$ is bounded and quasi-increasing. Must it also converge? If so prove it; if not, give a counter-example.

8. The **Alternating Series Test** (p. 74 of the text) says that if we have an alternating series $\sum(-1)^n b_n$ for which $\{b_n\}$ is a decreasing sequence of positive terms with $\lim b_n = 0$, then the sum converges.

Provide a proof of the Alternating Series Test.

Below are three ways to do this (choose one, choose them all, or use a different method, but make it clear what you are doing). Here $\{s_n\}$ denotes the sequence of partial sums given by $s_n = b_0 - b_1 + b_2 - b_3 + \ldots + (-1)^n b_n$.

- Show that $\{s_n\}$ is a Cauchy sequence.
- Use the Nested Intervals Theorem (p. 20).
- Consider the subsequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$, and apply the Monotone Convergence Theorem (p. 56).

9. Determine whether each of the following series converges. Fully justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n^2}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+2}$

(d) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \ldots$

10. Write a paragraph or so explaining how the Nested Intervals Theorem tells us that every (possibly infinite) decimal determines a unique real number (although some real numbers have more than one decimal expansion).