MAT511 homework, due Dec. 3, 2003

(1) Recall that in class (and in the handout copied from Eves; alternatively, a similar discussion can be found at `http://www.shu.edu/projects/reals/logic/numbers.html`), we considered the equivalence relation on $\mathbb{N} \times \mathbb{N}$ given by $(a, b) \sim (c, d)$ whenever $a+d = b+c$. We said that the set of equivalence classes corresponded to the integers $\mathbb{Z}$, where the each natural number n corresponds to equivalence class with elements of the form $(x+n, x)$ while negative integers correspond to classes of the form $(x, x+n)$.

Show that the relation $\preceq$ given by $(a, b) \preceq (c, d) \Leftrightarrow a+d \leq b+c$ defines a total order on the equivalence classes, which corresponds to the usual notion of order on $\mathbb{Z}$. (Recall that a total order is a partial order in which all elements are comparable.)

(2) If (a, b) and (c, d) are representatives of two equivalence classes as above, we can define multiplication as $(a, b) \cdot (c, d) = (ad + bc, ac + bd)$. Remember that these are equivalence classes, so the statement $(2, 1) \cdot (2, 1) = (4, 5)$ means $1 \cdot 1 = 1$.

Using this definition, show that if n and m are negative integers, $n \cdot m$ is a positive integer.

(3) We discussed how each real numbers corresponds to a Dedekind cut, or an infinite decimal that doesn't end in all 9s. Let $\mathcal{D}$ be the set of all real numbers greater than 0 and less than 1 which don't use the digits 1, 3, 5, 7, or 9 in their decimal expansion. Show that $\mathcal{D}$ is an uncountable set.

(4) Let $\mathcal{F}$ be the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. What is the cardinality of $\mathcal{F}$? Hint: You might find it conceptually easier to first think about the set $\mathcal{F}_{10}$ of all functions from $\mathbb{N}$ to $\{0, 1, 2, \ldots, 9\}$; $\mathcal{F}$ and $\mathcal{F}_{10}$ have the same cardinality.